5.1 The Natural Logarithmic Function: Differentiation

- Develop and use properties of the natural logarithmic function.
- Understand the definition of the number e.
- Find derivatives of functions involving the natural logarithmic function.

JOHN NAPIER (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Napier coined the term *logarithm*, from the two Greek words *logos* (or ratio) and *arithmos* (or number), to describe the theory that he spent 20 years developing and that first appeared in the book *Mirifici Logarithmorum canonis descriptio* (A Description of the Marvelous Rule of Logarithms). Although he did not introduce the *natural* logarithmic function, it is sometimes called the *Napierian* logarithm.

See LarsonCalculus.com to read more of this biography.

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

has an important disclaimer—it does not apply when $n = -1$. Consequently, yo have not yet found an antiderivative for the function $f(x) = 1/x$. In this section, yo will use the Second Fundamental Theorem of Calculus to *define* such a function. Thi antiderivative is a function that you have not encountered previously in the text. is neither algebraic nor trigonometric but falls into a new class of functions calle *logarithmic functions*. This particular function is the **natural logarithmic function.**

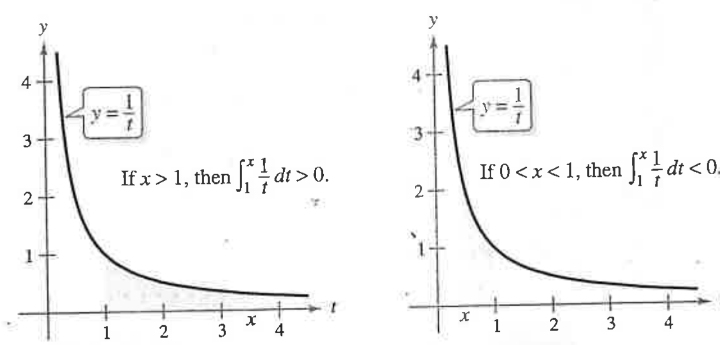
Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

From this definition, you can see that $\ln x$ is positive for $x > 1$ and negative f $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln 1 = 0$, because the upper and low limits of integration are equal when $x = 1$.



If $x > 1$, then $\ln x > 0$. If $0 < x < 1$, then $\ln x < 0$.
Figure 5.1

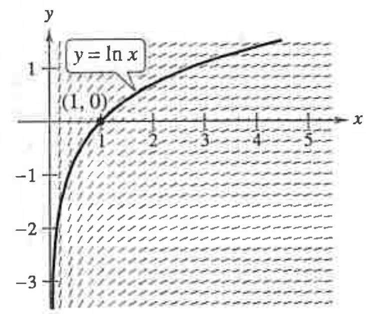
Exploration

Graphing the Natural Logarithmic Function Using *only* the definition of the natural logarithmic function, sketch a graph of the function. Explain your reasoning.

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

Figure 5.2 is a computer-generated graph, called a *slope field (or direction field)*, showing small line segments of slope $1/x$. The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$. (You will study slope fields in Section 6.1.)



Each small line segment has a slope of $\frac{1}{x}$.

Figure 5.2

THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Proof The domain of $f(x) = \ln x$ is $(0, \infty)$ by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

$$f'(x) = \frac{1}{x} \qquad \text{First derivative}$$

is positive for $x > 0$, as shown in Figure 5.3. It is concave downward because its second derivative

$$f''(x) = -\frac{1}{x^2} \qquad \text{Second derivative}$$

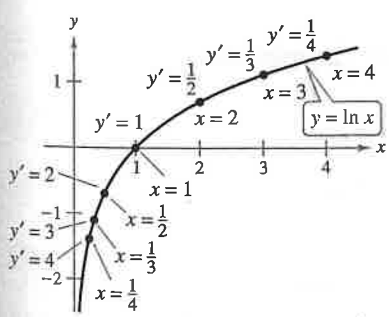
is negative for $x > 0$. The proof that f is one-to-one is given in Appendix A. The following limits imply that its range is the entire real number line.

$$\lim_{x\to 0^+} \ln x = -\infty$$

and

$$\lim_{x\to \infty} \ln x = \infty$$

Verification of these two limits is given in Appendix A. ■



The natural logarithmic function is increasing, and its graph is concave downward.

Figure 5.3

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that the properties listed on the next page are characteristic of all logarithms.

> **THEOREM 5.2 Logarithmic Properties**
>
> If a and b are positive numbers and n is rational, then the following properties are true.
>
> **1.** $\ln 1 = 0$ **2.** $\ln(ab) = \ln a + \ln b$
>
> **3.** $\ln(a^n) = n \ln a$ **4.** $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Proof The first property has already been discussed. The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant. From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, you know that

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

So, consider the two derivatives

$$\frac{d}{dx}[\ln(ax)] = \frac{a}{ax} = \frac{1}{x}$$

and

$$\frac{d}{dx}[\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

Because $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ at most by a constant, $\ln(ax) = \ln a + \ln x + C$. By letting $x = 1$, you can see that $C = 0$. The third property can be proved similarly by comparing the derivatives of $\ln(x^n)$ and $n \ln x$. Finally, using the second and third properties, you can prove the fourth property.

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b$$

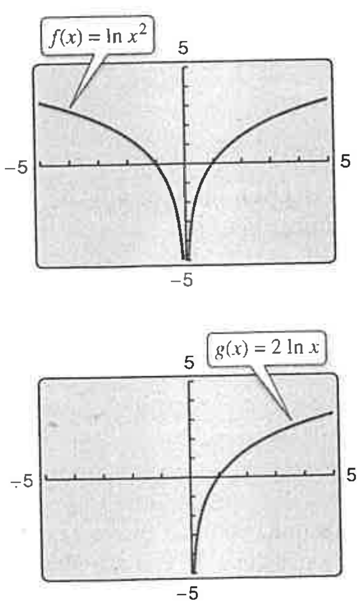


Figure 5.4

EXAMPLE 1 **Expanding Logarithmic Expressions**

a. $\ln \frac{10}{9} = \ln 10 - \ln 9$ Property 4

b. $\ln\sqrt{3x + 2} = \ln(3x + 2)^{1/2}$ Rewrite with rational exponent.

$= \frac{1}{2}\ln(3x + 2)$ Property 3

c. $\ln\frac{6x}{5} = \ln(6x) - \ln 5$ Property 4

$= \ln 6 + \ln x - \ln 5$ Property 2

d. $\ln\frac{(x^2 + 3)^2}{x\sqrt[3]{x^2 + 1}} = \ln(x^2 + 3)^2 - \ln\left(x\sqrt[3]{x^2 + 1}\right)$

$= 2\ln(x^2 + 3) - [\ln x + \ln(x^2 + 1)^{1/3}]$

$= 2\ln(x^2 + 3) - \ln x - \ln(x^2 + 1)^{1/3}$

$= 2\ln(x^2 + 3) - \ln x - \frac{1}{3}\ln(x^2 + 1)$

When using the properties of logarithms to rewrite logarithmic functions, you mu check to see whether the domain of the rewritten function is the same as the domain the original. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x =$ and the domain of $g(x) = 2 \ln x$ is all positive real numbers. (See Figure 5.4.)

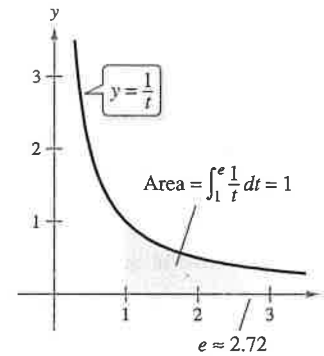
The Number e

THE NUMBER e

The symbol e was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 and therefore $\log_{10} 10 = 1$. (You will learn more about this in Section 5.5.)

The **base for the natural logarithm** is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5. This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.



e is the base for the natural logarithm because $\ln e = 1$.
Figure 5.5

$$e \approx 2.71828182846$$

Definition of e

The letter e denotes the positive real number such that

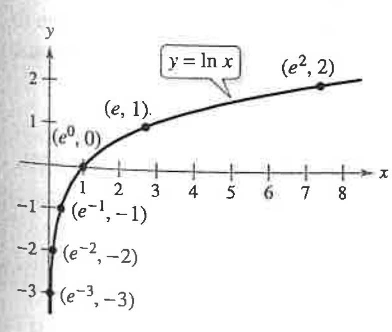
$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

FOR FURTHER INFORMATION To learn more about the number e, see the article "Unexpected Occurrences of the Number e" by Harris S. Shultz and Bill Leonard in *Mathematics Magazine*. To view this article, go to *MathArticles.com.*

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned}\ln(e^n) &= n \ln e \\ &= n(1) \\ &= n\end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of n, as shown in the table and in Figure 5.6.



If $x = e^n$, then $\ln x = n$.
Figure 5.6

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2

The logarithms shown in the table above are convenient because the x-values are integer powers of e. Most logarithmic expressions are, however, best evaluated with a calculator.

EXAMPLE 2 **Evaluating Natural Logarithmic Expressions**

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

EXAMPLE 3 Differentiation of Logarithmic Functions

See LarsonCalculus.com for an interactive version of this type of example.

a. $\frac{d}{dx}[\ln 2x] = \frac{u'}{u} = \frac{2}{2x} = \frac{1}{x}$ $u = 2x$

b. $\frac{d}{dx}[\ln(x^2 + 1)] = \frac{u'}{u} = \frac{2x}{x^2 + 1}$ $u = x^2 + 1$

c. $\frac{d}{dx}[x \ln x] = x\left(\frac{d}{dx}[\ln x]\right) + (\ln x)\left(\frac{d}{dx}[x]\right)$ Product Rule

$= x\left(\frac{1}{x}\right) + (\ln x)(1)$

$= 1 + \ln x$

d. $\frac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \frac{d}{dx}[\ln x]$ Chain Rule

$= 3(\ln x)^2 \frac{1}{x}$

Napier used logarithmic properties to simplify *calculations* involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify *differentiation* involving products, quotients, and powers.

EXAMPLE 4 Logarithmic Properties as Aids to Differentiation

Differentiate

$$f(x) = \ln\sqrt{x + 1}.$$

Solution Because

$$f(x) = \ln\sqrt{x + 1} = \ln(x + 1)^{1/2} = \frac{1}{2}\ln(x + 1)$$ Rewrite before differentiating.

you can write

$$f'(x) = \frac{1}{2}\left(\frac{1}{x + 1}\right) = \frac{1}{2(x + 1)}.$$ Differentiate.

EXAMPLE 5 **Logarithmic Properties as Aids to Differentiation**

Differentiate $f(x) = \ln \dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

Solution Because

$$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \qquad \text{Write original function.}$$

$$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) \qquad \text{Rewrite before differentiating.}$$

you can write

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) \qquad \text{Differentiate.}$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}. \qquad \text{Simplify.}$$

■

In Examples 4 and 5, be sure you see the benefit of applying logarithmic properties *before* differentiating. Consider, for instance, the difficulty of direct differentiation of the function given in Example 5.

On occasion, it is convenient to use logarithms as aids in differentiating *nonlogarithmic* functions. This procedure is called **logarithmic differentiation.** In general, use logarithmic differentiation when differentiating (1) a function involving many factors or (2) a function having both a variable base and a variable exponent [see Section 5.5, Example 5(d)].

EXAMPLE 6 **Logarithmic Differentiation**

Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$

Solution Note that $y > 0$ for all $x \neq 2$. So, $\ln y$ is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2 \qquad \text{Write original equation.}$$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take natural log of each side.}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate.}$$

$$\frac{y'}{y} = \frac{x^2+2x+2}{(x-2)(x^2+1)} \qquad \text{Simplify.}$$

$$y' = y\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Solve for } y'.$$

$$y' = \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y.$$

$$y' = \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify.}$$

■

REMARK You could also solve the problem in Example 6 without using logarithmic differentiation by using the Power and Quotient Rules. Use these rules to find the derivative and show that the result is equivalent to the one in Example 6. Which method do you prefer?

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The next theorem states that you can differentiate functions of the form $y = \ln|u|$ as though the absolute value notation was not present.

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Proof If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u < 0$, then $|u| = -u$, and you have

$$\begin{aligned}\frac{d}{dx}[\ln|u|] &= \frac{d}{dx}[\ln(-u)]\\ &= \frac{-u'}{-u}\\ &= \frac{u'}{u}.\end{aligned}$$

EXAMPLE 7 Derivative Involving Absolute Value

Find the derivative of

$$f(x) = \ln|\cos x|.$$

Solution Using Theorem 5.4, let $u = \cos x$ and write

$$\begin{aligned}\frac{d}{dx}[\ln|\cos x|] &= \frac{u'}{u} && \frac{d}{dx}[\ln|u|] = \frac{u'}{u}\\ &= \frac{-\sin x}{\cos x} && u = \cos x\\ &= -\tan x. && \text{Simplify.}\end{aligned}$$

EXAMPLE 8 Finding Relative Extrema

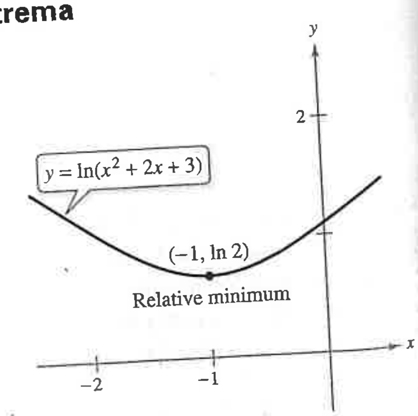
Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$

Solution Differentiating y, you obtain

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 3}.$$

Because $dy/dx = 0$ when $x = -1$, you can apply the First Derivative Test and conclude that a relative minimum occurs at the point $(-1, \ln 2)$. Because there are no other critical points, it follows that this is the only relative extremum, as shown in the figure.



The derivative of y changes from negative to positive at $x = -1$.

5.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Natural Logarithmic Function** Explain why $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$.

2. **Logarithmic Properties** What is the value of n?

$$\ln 4 + \ln(n^{-1}) = \ln 4 - \ln 7$$

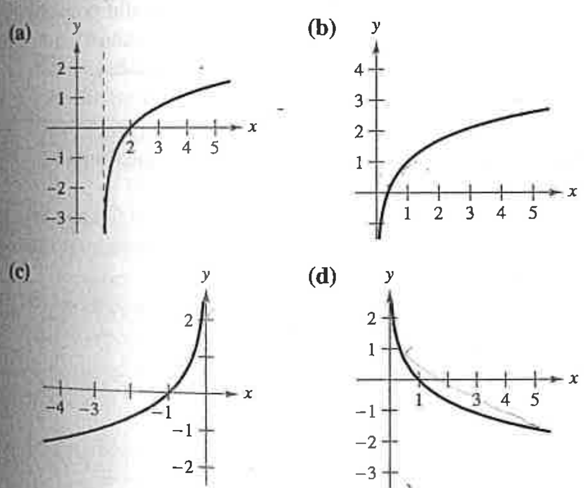
3. **The Number e** How is the number e defined?

4. **Differentiation of Logarithmic Functions** State the Chain Rule version of the derivative of the natural logarithmic function in your own words.

Evaluating a Logarithm Using Technology **In Exercises 5–8, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral $\int_1^x (1/t)\,dt$.**

5. $\ln 45$
6. $\ln 8.3$
7. $\ln 0.8$
8. $\ln 0.6$

Matching **In Exercises 9–12, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**



9. $f(x) = \ln x + 1$
10. $f(x) = -\ln x$
11. $f(x) = \ln(x - 1)$
12. $f(x) = -\ln(-x)$

Sketching a Graph **In Exercises 13–18, sketch the graph of the function and state its domain.**

13. $f(x) = 3 \ln x$
14. $f(x) = -2 \ln x$
15. $f(x) = \ln 2x$
16. $f(x) = \ln|x|$
17. $f(x) = \ln(x - 3)$
18. $f(x) = \ln x - 4$

Using Properties of Logarithms **In Exercises 19 and 20, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.**

19. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$
20. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

Expanding a Logarithmic Expression **In Exercises 21–30, use the properties of logarithms to expand the logarithmic expression.**

21. $\ln \dfrac{x}{4}$
22. $\ln \sqrt{x^5}$
23. $\ln \dfrac{xy}{z}$
24. $\ln(xyz)$
25. $\ln\left(x\sqrt{x^2 + 5}\right)$
26. $x \ln \sqrt{x - 4}$
27. $\ln \sqrt{\dfrac{x - 1}{x}}$
28. $\ln(3e^2)$
29. $\ln z(z - 1)^2$
30. $\ln \dfrac{z}{e}$

Condensing a Logarithmic Expression **In Exercises 31–36, write the expression as a logarithm of a single quantity.**

31. $\ln(x - 2) - \ln(x + 2)$
32. $3 \ln x + 2 \ln y - 4 \ln z$
33. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$
34. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$
35. $4 \ln 2 - \frac{1}{2} \ln(x^3 + 6x)$
36. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

Verifying Properties of Logarithms **In Exercises 37 and 38, (a) verify that $f = g$ by using a graphing utility to graph f and g in the same viewing window and (b) verify that $f = g$ algebraically.**

37. $f(x) = \ln \dfrac{x^2}{4}, \quad x > 0, \quad g(x) = 2 \ln x - \ln 4$
38. $f(x) = \ln \sqrt{x(x^2 + 1)}, \quad g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

Finding a Limit **In Exercises 39–42, find the limit.**

39. $\lim_{x \to 3^+} \ln(x - 3)$
40. $\lim_{x \to 6^-} \ln(6 - x)$
41. $\lim_{x \to 2^-} \ln[x^2(3 - x)]$
42. $\lim_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}}$

Finding a Derivative **In Exercises 43–66, find the derivative of the function.**

43. $f(x) = \ln 3x$
44. $f(x) = \ln(x - 1)$
45. $f(x) = \ln(x^2 + 3)$
46. $h(x) = \ln(2x^2 + 1)$
47. $y = (\ln x)^4$
48. $y = x^2 \ln x$
49. $y = \ln(t + 1)^2$
50. $y = \ln \sqrt{x^2 - 4}$

51. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

52. $y = \ln[t(t^2 + 3)^3]$

53. $f(x) = \ln \dfrac{x}{x^2 + 1}$

54. $f(x) = \ln \dfrac{2x}{x + 3}$

55. $g(t) = \dfrac{\ln t}{t^2}$

56. $h(t) = \dfrac{\ln t}{t^3 + 5}$

57. $y = \ln(\ln x^2)$

58. $y = \ln(\ln x)$

59. $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$

60. $y = \ln \sqrt[3]{\dfrac{x - 1}{x + 1}}$

61. $f(x) = \ln \dfrac{\sqrt{4 + x^2}}{x}$

62. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

63. $y = \ln|\sin x|$

64. $y = \ln|\csc x|$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

66. $y = \ln|\sec x + \tan x|$

Finding an Equation of a Tangent Line In Exercises 67–74, (a) find an equation of the tangent line to the graph of the function at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *tangent* feature of a graphing utility to confirm your results.

67. $y = \ln x^4, \quad (1, 0)$

68. $y = \ln x^{2/3}, \quad (-1, 0)$

69. $f(x) = 3x^2 - \ln x, \quad (1, 3)$

70. $f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

71. $f(x) = \ln\sqrt{1 + \sin^2 x}, \quad \left(\dfrac{\pi}{4}, \ln\sqrt{\dfrac{3}{2}}\right)$

72. $f(x) = \sin 2x \ln x^2, \quad (1, 0)$

73. $y = x^3 \ln x^4, \quad (-1, 0)$

74. $f(x) = \dfrac{1}{2}x \ln x^2, \quad (-1, 0)$

Logarithmic Differentiation In Exercises 75–80, use logarithmic differentiation to find dy/dx.

75. $y = x\sqrt{x^2 + 1}, \quad x > 0$

76. $y = \sqrt{x^2(x + 1)(x + 2)}, \quad x > 0$

77. $y = \dfrac{x^2\sqrt{3x - 2}}{(x + 1)^2}, \quad x > \dfrac{2}{3}$

78. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}, \quad x > 1$

79. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}, \quad x > 1$

80. $y = \dfrac{(x + 1)(x - 2)}{(x - 1)(x + 2)}, \quad x > 2$

Implicit Differentiation In Exercises 81–84, use implicit differentiation to find dy/dx.

81. $x^2 - 3 \ln y + y^2 = 10$

82. $\ln xy + 5x = 30$

83. $4x^3 + \ln y^2 + 2y = 2x$

84. $4xy + \ln x^2 y = 7$

Differential Equation In Exercises 85 and 86, verify that the function is a solution of the differential equation.

Function	Differential Equation
85. $y = 2 \ln x + 3$	$xy'' + y' = 0$
86. $y = x \ln x - 4x$	$x + y - xy' = 0$

Relative Extrema and Points of Inflection In Exercises 87–92, locate any relative extrema and points of inflection. Use a graphing utility to confirm your results.

87. $y = \dfrac{x^2}{2} - \ln x$

88. $y = 2x - \ln 2x$

89. $y = x \ln x$

90. $y = \dfrac{\ln x}{x}$

91. $y = \dfrac{x}{\ln x}$

92. $y = x^2 \ln \dfrac{x}{4}$

Using Newton's Method In Exercises 93 and 94, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.

93. $y = \ln x, \quad y = -x$

94. $y = \ln x, \quad y = 3 - x$

EXPLORING CONCEPTS

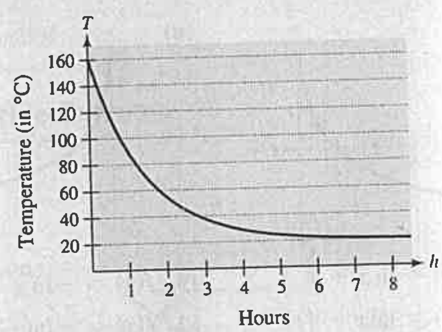
Comparing Functions In Exercises 95 and 96, let f be a function that is positive and differentiable on the entire real number line and let $g(x) = \ln f(x)$.

95. When g is increasing, must f be increasing? Explain.

96. When the graph of f is concave upward, must the graph of g be concave upward? Explain.

97. **Think About It** Is $\ln xy = \ln x \ln y$ a valid property of logarithms, where $x > 0$ and $y > 0$? Explain.

98. **HOW DO YOU SEE IT?** The graph shows the temperature T (in degrees Celsius) of an object h hours after it is removed from a furnace.



(a) Find $\lim_{h\to\infty} T$. What does this limit represent?

(b) When is the temperature changing most rapidly?

True or False? In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

99. $\ln(a^{n+m}) = n \ln a + m \ln a$, where $a > 0$ and m and n are rational.

100. $\dfrac{d}{dx}[\ln(cx)] = \dfrac{d}{dx}[\ln x]$, where $c > 0$

101. If $y = \ln \pi$, then $y' = 1/\pi$.

102. If $y = \ln e$, then $y' = 1$.

103. Home Mortgage The term t (in years) of a \$200,000 home mortgage at 7.5% interest can be approximated by

$$t = 13.375 \ln\left(\frac{x}{x - 1250}\right), \quad x > 1250$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1398.43. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1611.19. What is the total amount paid?

(d) Find the instantaneous rates of change of t with respect to x when $x = \$1398.43$ and $x = \$1611.19$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

104. Sound Intensity

The relationship between the number of decibels β and the intensity of a sound I in watts per centimeter squared is

$$\beta = \frac{10}{\ln 10} \ln\left(\frac{I}{10^{-16}}\right).$$

(a) Use the properties of logarithms to write the formula in simpler form.

(b) Determine the number of decibels of a sound with an intensity of 10^{-5} watt per square centimeter.

105. Modeling Data The table shows the temperatures T (in degrees Fahrenheit) at which water boils at selected pressures p (in pounds per square inch). (*Source: Standard Handbook of Mechanical Engineers*)

p	5	10	14.696 (1 atm)	20
T	162.24	193.21	212.00	227.96

p	30	40	60	80	100
T	250.33	267.25	292.71	312.03	327.81

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rates of change of T with respect to p when $p = 10$ and $p = 70$.

(c) Use a graphing utility to graph T'. Find $\lim_{p \to \infty} T'(p)$ and interpret the result in the context of the problem.

106. Modeling Data The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the model from part (b).

(d) Use the model to estimate the altitude when $p = 0.75$.

(e) Use the model to estimate the pressure when $h = 13$.

(f) Use the model to find the rates of change of pressure when $h = 5$ and $h = 20$. Interpret the results.

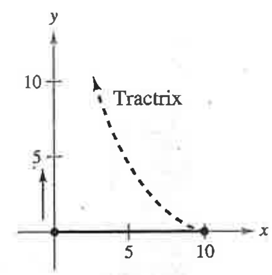
107. Tractrix A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What are the slopes of this path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as x approaches 10 from the left?



108. Prime Number Theorem There are 25 prime numbers less than 100. The **Prime Number Theorem** states that the number of primes less than x approaches

$$p(x) \approx \frac{x}{\ln x}.$$

Use this approximation to estimate the rate (in primes per 100 integers) at which the prime numbers occur when

(a) $x = 1000$.

(b) $x = 1{,}000{,}000$.

(c) $x = 1{,}000{,}000{,}000$.

109. Conjecture Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the greater rate for large values of x. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x, \quad g(x) = \sqrt{x}$

(b) $f(x) = \ln x, \quad g(x) = \sqrt[4]{x}$