P.1 Graphs and Models

- **Sketch the graph of an equation.**
- **Find the intercepts of a graph.**
- **Test a graph for symmetry with respect to an axis and the origin.**
- **Find the points of intersection of two graphs.**
- **Interpret mathematical models for real-life data.**

RENÉ DESCARTES (1596–1650)

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book *La Géométrie*, published in 1637.

See LarsonCalculus.com to read more of this biography.

The Graph of an Equation

In 1637, the French mathematician René Descartes revolutionized the study of mathematics by combining its two major fields—algebra and geometry. With Descartes's coordinate plane, geometric concepts could be formulated analytically and algebraic concepts could be viewed graphically. The power of this approach was such that within a century of its introduction, much of calculus had been developed.

The same approach can be followed in your study of calculus. That is, by viewing calculus from multiple perspectives—*graphically*, *analytically*, and *numerically*—you will increase your understanding of core concepts.

Consider the equation $3x + y = 7$. The point $(2, 1)$ is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as $(1, 4)$ and $(0, 7)$. To find other solutions systematically, solve the original equation for y.

$$y = 7 - 3x \qquad \text{Analytic approach}$$

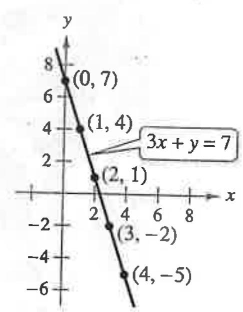
Then construct a **table of values** by substituting several values of x.

x	0	1	2	3	4
y	7	4	1	-2	-5

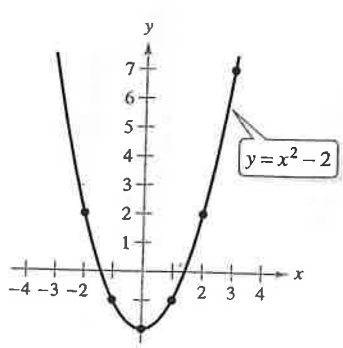
Numerical approach

From the table, you can see that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solutions of the original equation $3x + y = 7$. Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure P.1. Note that the sketch shown in Figure P.1 is referred to as the graph of $3x + y = 7$, even though it really represents only a *portion* of the graph. The entire graph would extend beyond the page.

In this course, you will study many sketching techniques. The simplest is point plotting—that is, you plot points until the basic shape of the graph seems apparent.



Graphical approach: $3x + y = 7$
Figure P.1



The parabola $y = x^2 - 2$
Figure P.2

EXAMPLE 1 Sketching a Graph by Point Plotting

To sketch the graph of $y = x^2 - 2$, first construct a table of values. Next, plot the points shown in the table. Then connect the points with a smooth curve, as shown in Figure P.2. This graph is a **parabola.** It is one of the conics you will study in Chapter 10.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7

■

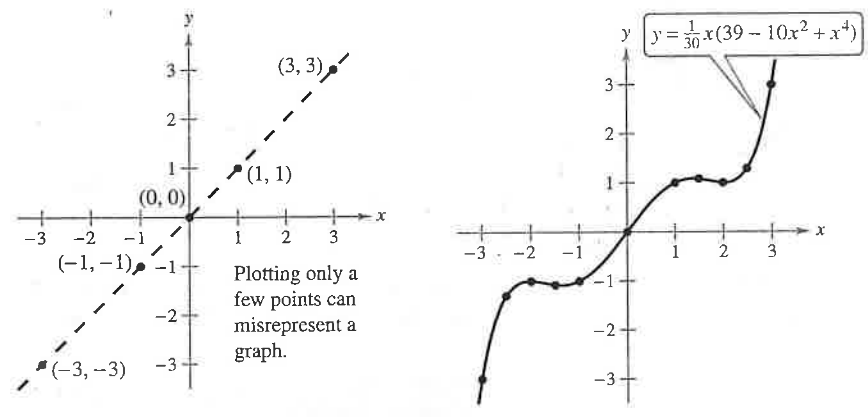
One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could badly misrepresent the graph. For instance, to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plot five points:

$$(-3, -3),\quad (-1, -1),\quad (0, 0),\quad (1, 1),\quad \text{and} \quad (3, 3)$$

as shown in Figure P.3(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure P.3(b).



(a) (b)
Figure P.3

Exploration

Comparing Graphical and Analytic Approaches
Use a graphing utility to graph each equation. In each case, find a viewing window that shows the important characteristics of the graph.

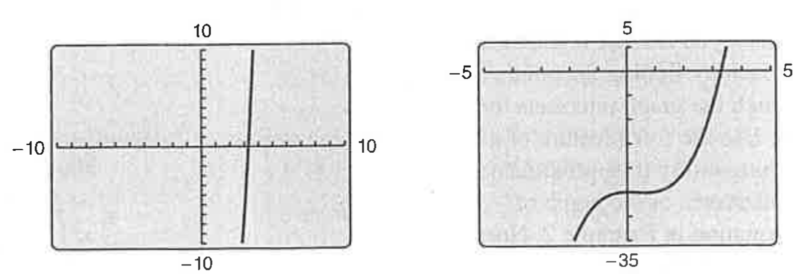
a. $y = x^3 - 3x^2 + 2x + 5$
b. $y = x^3 - 3x^2 + 2x + 25$
c. $y = -x^3 - 3x^2 + 20x + 5$
d. $y = 3x^3 - 40x^2 + 50x - 45$
e. $y = -(x + 12)^3$
f. $y = (x - 2)(x - 4)(x - 6)$

A purely graphical approach to this problem would involve a simple "guess, check, and revise" strategy. What types of things do you think an analytic approach might involve? For instance, does the graph have symmetry? Does the graph have turns? If so, where are they? As you proceed through Chapters 1, 2, and 3 of this text, you will study many new analytic tools that will help you analyze graphs of equations such as these.

TECHNOLOGY Graphing an equation has been made easier by technology. Even with technology, however, it is possible to misrepresent a graph badly. For instance, each of the graphing utility* screens in Figure P.4 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. So, whether you are sketching a graph by hand or using a graphing utility, you must realize that different "viewing windows" can produce very different views of a graph. In choosing a viewing window, your goal is to show a view of the graph that fits well in the context of the problem.



Graphing utility screens of $y = x^3 - x^2 - 25$
Figure P.4

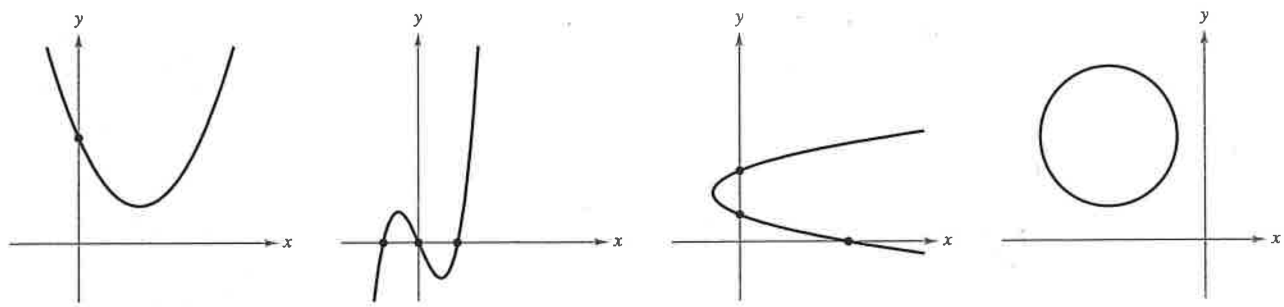
*In this text, the term *graphing utility* means either a graphing calculator, such as the *TI-Nspire*, or computer graphing software, such as *Maple* or *Mathematica*.

Intercepts of a Graph

REMARK Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, when the term *intercept* is used in this text, it will mean either the point or the coordinate.

Two types of solution points that are especially useful in graphing an equation are those having zero as their x- or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. The point $(a, 0)$ is an **x-intercept** of the graph of an equation when it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a **y-intercept** of the graph of an equation when it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure P.5.



No x-intercepts
One y-intercept

Three x-intercepts
One y-intercept

One x-intercept
Two y-intercepts

No intercepts

Figure P.5

EXAMPLE 2 Finding x- and y-Intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x-intercepts, let y be zero and solve for x.

$$x^3 - 4x = 0 \qquad \text{Let } y \text{ be zero.}$$

$$x(x - 2)(x + 2) = 0 \qquad \text{Factor.}$$

$$x = 0, 2, \text{ or } -2 \qquad \text{Solve for } x.$$

Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

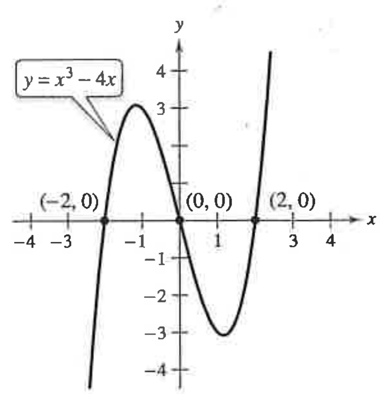
$$(0, 0), \quad (2, 0), \quad \text{and} \quad (-2, 0). \qquad x\text{-intercepts}$$

To find the y-intercepts, let x be zero. Doing this produces $y = 0$. So, the y-intercept is

$$(0, 0). \qquad y\text{-intercept}$$

(See Figure P.6.)

TECHNOLOGY Example 2 uses an analytic approach to finding intercepts. When an analytic approach is not possible, you can use a graphical approach by finding the points at which the graph intersects the axes. Use the *trace* feature of a graphing utility to approximate the intercepts of the graph of the equation in Example 2. Note that your utility may have a built-in program that can find the x-intercepts of a graph. (Your utility may call this the *root* or *zero* feature.) If so, use the program to find the x-intercepts of the graph of the equation in Example 2.



Intercepts of a graph
Figure P.6

Symmetry of a Graph

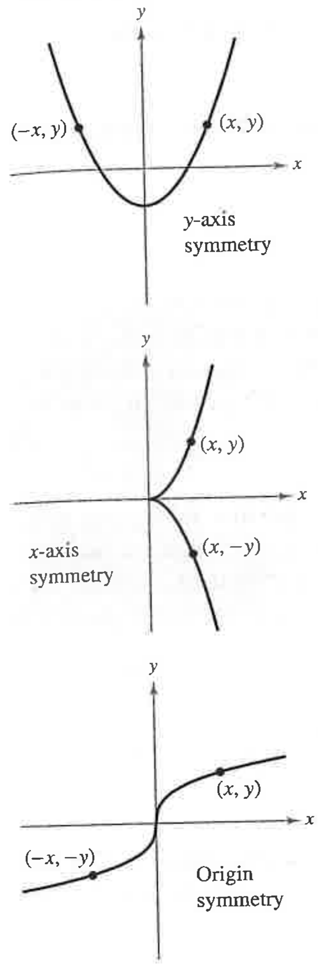


Figure P.7

Knowing the symmetry of a graph before attempting to sketch it is useful because you need only half as many points to sketch the graph. The three types of symmetry listed below can be used to help sketch the graphs of equations (see Figure P.7).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, then $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, then $(x, -y)$ is also a point on the graph. This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, then $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis when replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis when replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin when replacing x by $-x$ and y by $-y$ yields an equivalent equation.

The graph of a polynomial has symmetry with respect to the y-axis when each term has an even exponent (or is a constant). For instance, the graph of

$$y = 2x^4 - x^2 + 2$$

has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin when each term has an odd exponent, as illustrated in Example 3.

EXAMPLE 3 Testing for Symmetry

Test the graph of $y = 2x^3 - x$ for symmetry with respect to (a) the y-axis and (b) the origin.

Solution

a.

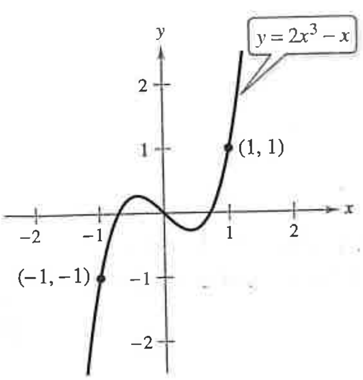
$y = 2x^3 - x$	Write original equation.
$y = 2(-x)^3 - (-x)$	Replace x by $-x$.
$y = -2x^3 + x$	Simplify. The result is *not* an equivalent equation.

Because replacing x by $-x$ does *not* yield an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is *not* symmetric with respect to the y-axis.

b.

$y = 2x^3 - x$	Write original equation.
$-y = 2(-x)^3 - (-x)$	Replace x by $-x$ and y by $-y$.
$-y = -2x^3 + x$	Simplify.
$y = 2x^3 - x$	Equivalent equation

Because replacing x by $-x$ and y by $-y$ yields an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure P.8. ■



Origin symmetry
Figure P.8

EXAMPLE 4 **Using Intercepts and Symmetry to Sketch a Graph**

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graph of $x - y^2 = 1$.

Solution The graph is symmetric with respect to the x-axis because replacing y by $-y$ yields an equivalent equation.

$$x - y^2 = 1 \quad \text{Write original equation.}$$
$$x - (-y)^2 = 1 \quad \text{Replace } y \text{ by } -y.$$
$$x - y^2 = 1 \quad \text{Equivalent equation}$$

This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis. To sketch the graph, first plot the x-intercept and the points above the x-axis. Then reflect in the x-axis to obtain the entire graph, as shown in Figure P.9. ■

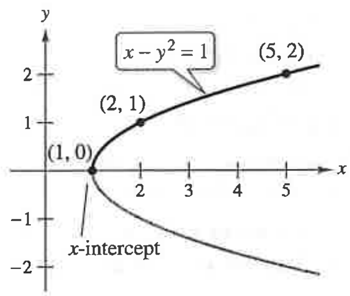


Figure P.9

TECHNOLOGY Graphing utilities are designed so that they most easily graph equations in which y is a function of x (see Section P.3 for a definition of *function*). To graph other types of equations, you need to split the graph into two or more parts *or* you need to use a different graphing mode. For instance, to graph the equation in Example 4, you can split it into two parts.

$$y_1 = \sqrt{x - 1} \quad \text{Top portion of graph}$$
$$y_2 = -\sqrt{x - 1} \quad \text{Bottom portion of graph}$$

Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 **Finding Points of Intersection**

Find all points of intersection of the graphs of

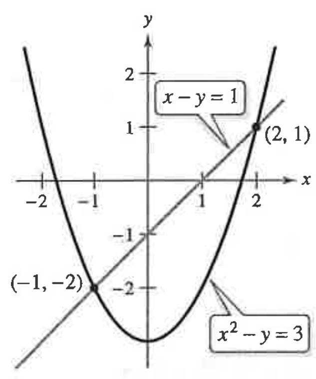
$$x^2 - y = 3 \quad \text{and} \quad x - y = 1.$$

Solution Begin by sketching the graphs of both equations in the *same* rectangular coordinate system, as shown in Figure P.10. From the figure, it appears that the graphs have two points of intersection. You can find these two points as follows.

$$y = x^2 - 3 \quad \text{Solve first equation for } y.$$
$$y = x - 1 \quad \text{Solve second equation for } y.$$
$$x^2 - 3 = x - 1 \quad \text{Equate } y\text{-values.}$$
$$x^2 - x - 2 = 0 \quad \text{Write in general form.}$$
$$(x - 2)(x + 1) = 0 \quad \text{Factor.}$$
$$x = 2 \text{ or } -1 \quad \text{Solve for } x.$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \quad \text{and} \quad (-1, -2). \quad \text{Points of intersection}$$ ■



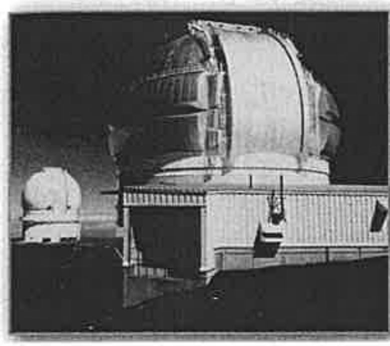
Two points of intersection
Figure P.10

You can check the points of intersection in Example 5 by substituting into *both* of the original equations or by using the *intersect* feature of a graphing utility.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models.** In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals—accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Appendix G explores these goals more completely.

EXAMPLE 6 Comparing Two Mathematical Models



The Mauna Loa Observatory in Hawaii has been measuring the increasing concentration of carbon dioxide in Earth's atmosphere since 1958.

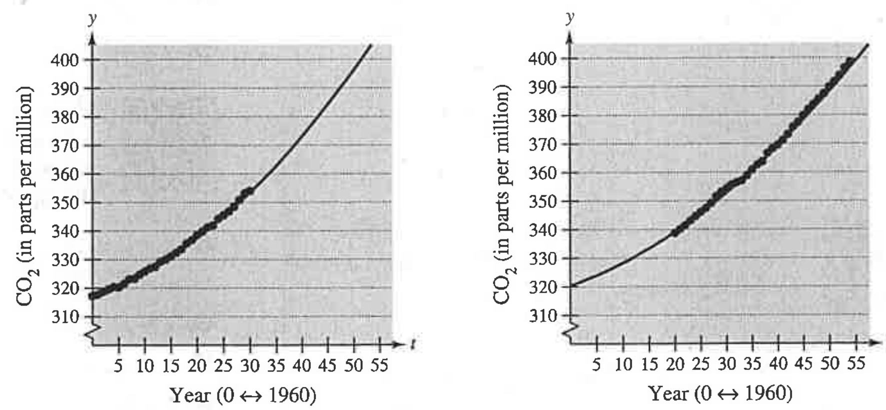
The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration y (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure P.11. In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035, using the quadratic model

$$y = 0.018t^2 + 0.70t + 316.2 \qquad \text{Quadratic model for 1960–1990 data}$$

where $t = 0$ represents 1960, as shown in Figure P.11(a). The data shown in Figure P.11(b) represent the years 1980 through 2014 and can be modeled by

$$y = 0.014t^2 + 0.66t + 320.3 \qquad \text{Quadratic model for 1980–2014 data}$$

where $t = 0$ represents 1960. What was the prediction given in the *Scientific American* article in 1990? Given the second model for 1980 through 2014, does this prediction for the year 2035 seem accurate?



(a) (b)

Figure P.11

Solution To answer the first question, substitute $t = 75$ (for 2035) into the first model.

$$y = 0.018(75)^2 + 0.70(75) + 316.2 = 469.95 \qquad \text{Model for 1960–1990 data}$$

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035. Using the model for the 1980–2014 data, the prediction for the year 2035 is

$$y = 0.014(75)^2 + 0.66(75) + 320.3 = 448.55. \qquad \text{Model for 1980–2014 data}$$

So, based on the model for 1980–2014, it appears that the 1990 prediction was too high. ■

The models in Example 6 were developed using a procedure called *least squares regression* (see Section 13.9). The older model has a correlation of $r^2 \approx 0.997$, and for the newer model it is $r^2 \approx 0.999$. The closer r^2 is to 1, the "better" the model.

P.1 Exercises

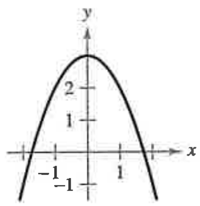
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

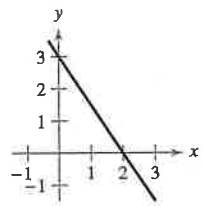
1. **Finding Intercepts** Describe how to find the x- and y-intercepts of the graph of an equation.
2. **Verifying Points of Intersection** How can you check that an ordered pair is a point of intersection of two graphs?

Matching **In Exercises 3–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

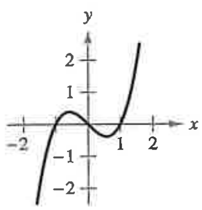
(a)



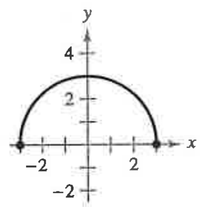
(b)



(c)



(d)



3. $y = -\frac{3}{2}x + 3$
4. $y = \sqrt{9 - x^2}$
5. $y = 3 - x^2$
6. $y = x^3 - x$

Sketching a Graph by Point Plotting **In Exercises 7–16, sketch the graph of the equation by point plotting.**

7. $y = \frac{1}{2}x + 2$
8. $y = 5 - 2x$
9. $y = 4 - x^2$
10. $y = (x - 3)^2$
11. $y = |x + 1|$
12. $y = |x| - 1$
13. $y = \sqrt{x} - 6$
14. $y = \sqrt{x + 2}$
15. $y = \dfrac{3}{x}$
16. $y = \dfrac{1}{x + 2}$

Approximating Solution Points Using Technology **In Exercises 17 and 18, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places.**

17. $y = \sqrt{5 - x}$
    (a) $(2, y)$
    (b) $(x, 3)$
18. $y = x^5 - 5x$
    (a) $(-0.5, y)$
    (b) $(x, -4)$

The symbol and a red exercise number indicates that a video solution can be seen at *CalcView.com*.

Finding Intercepts **In Exercises 19–28, find any intercepts.**

19. $y = 2x - 5$
20. $y = 4x^2 + 3$
21. $y = x^2 + x - 2$
22. $y^2 = x^3 - 4x$
23. $y = x\sqrt{16 - x^2}$
24. $y = (x - 1)\sqrt{x^2 + 1}$
25. $y = \dfrac{2 - \sqrt{x}}{5x + 1}$
26. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$
27. $x^2y - x^2 + 4y = 0$
28. $y = 2x - \sqrt{x^2 + 1}$

Testing for Symmetry **In Exercises 29–40, test for symmetry with respect to each axis and to the origin.**

29. $y = x^2 - 6$
30. $y = 9x - x^2$
31. $y^2 = x^3 - 8x$
32. $y = x^3 + x$
33. $xy = 4$
34. $xy^2 = -10$
35. $y = 4 - \sqrt{x + 3}$
36. $xy - \sqrt{4 - x^2} = 0$
37. $y = \dfrac{x}{x^2 + 1}$
38. $y = \dfrac{x^5}{4 - x^2}$
39. $y = |x^3 + x|$
40. $|y| - x = 3$

Using Intercepts and Symmetry to Sketch a Graph **In Exercises 41–56, find any intercepts and test for symmetry. Then sketch the graph of the equation.**

41. $y = 2 - 3x$
42. $y = \frac{2}{3}x + 1$
43. $y = 9 - x^2$
44. $y = 2x^2 + x$
45. $y = x^3 + 2$
46. $y = x^3 - 4x$
47. $y = x\sqrt{x + 5}$
48. $y = \sqrt{25 - x^2}$
49. $x = y^3$
50. $x = y^4 - 16$
51. $y = \dfrac{8}{x}$
52. $y = \dfrac{10}{x^2 + 1}$
53. $y = 6 - |x|$
54. $y = |6 - x|$
55. $3y^2 - x = 9$
56. $x^2 + 4y^2 = 4$

Finding Points of Intersection **In Exercises 57–62, find the points of intersection of the graphs of the equations.**

57. $x + y = 8$
    $4x - y = 7$
58. $3x - 2y = -4$
    $4x + 2y = -10$
59. $x^2 + y = 15$
    $-3x + y = 11$
60. $x = 3 - y^2$
    $y = x - 1$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by the use of appropriate technology.

61. $x^2 + y^2 = 5$
 $x - y = 1$

62. $x^2 + y^2 = 16$
 $x + 2y = 4$

Finding Points of Intersection Using Technology **In Exercises 63–66, use a graphing utility to find the points of intersection of the graphs of the equations. Check your results analytically.**

63. $y = x^3 - 2x^2 + x - 1$
 $y = -x^2 + 3x - 1$

64. $y = x^4 - 2x^2 + 1$
 $y = 1 - x^2$

65. $y = \sqrt{x + 6}$
 $y = \sqrt{-x^2 - 4x}$

66. $y = -|2x - 3| + 6$
 $y = 6 - x$

67. **Modeling Data** The table shows the Gross Domestic Product, or GDP (in trillions of dollars), for 2009 through 2014. *(Source: U.S. Bureau of Economic Analysis)*

Year	2009	2010	2011	2012	2013	2014
GDP	14.4	15.0	15.5	16.2	16.7	17.3

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at + b$ for the data. In the model, y represents the GDP (in trillions of dollars) and t represents the year, with $t = 9$ corresponding to 2009.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the GDP in the year 2024.

68. **Modeling Data**

The table shows the numbers of cell phone subscribers (in millions) in the United States for selected years. *(Source: CTIA-The Wireless Association)*

Year	2000	2002	2004	2006
Number	109	141	182	233

Year	2008	2010	2012	2014
Number	270	296	326	355

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the number of subscribers (in millions) and t represents the year, with $t = 0$ corresponding to 2000.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the number of cell phone subscribers in the United States in the year 2024.

ChrisMilesPhoto/Shutterstock.com

69. **Break-Even Point** Find the sales necessary to break even ($R = C$) when the cost C of producing x units is $C = 2.04x + 5600$ and the revenue R from selling x units is $R = 3.29x$.

70. **Using Solution Points** For what values of k does the graph of $y^2 = 4kx$ pass through the point?

(a) $(1, 1)$ (b) $(2, 4)$
(c) $(0, 0)$ (d) $(3, 3)$

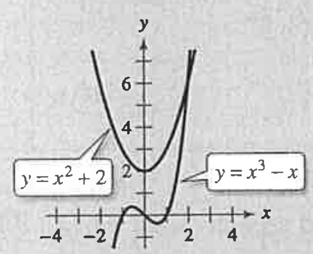
EXPLORING CONCEPTS

71. **Using Intercepts** Write an equation whose graph has intercepts at $x = -\frac{3}{2}$, $x = 4$, and $x = \frac{5}{2}$. (There is more than one correct answer.)

72. **Symmetry** A graph is symmetric with respect to the x-axis and to the y-axis. Is the graph also symmetric with respect to the origin? Explain.

73. **Symmetry** A graph is symmetric with respect to one axis and to the origin. Is the graph also symmetric with respect to the other axis? Explain.

74. **HOW DO YOU SEE IT?** Use the graphs of the two equations to answer the questions below.



(a) What are the intercepts for each equation?

(b) Determine the symmetry for each equation.

(c) Determine the point of intersection of the two equations.

True or False? **In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the x-axis, then $(4, -5)$ is also a point on the graph.

76. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the y-axis, then $(4, -5)$ is also a point on the graph.

77. If $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of

$$y = ax^2 + bx + c$$

has two x-intercepts.

78. If $b^2 - 4ac = 0$ and $a \neq 0$, then the graph of

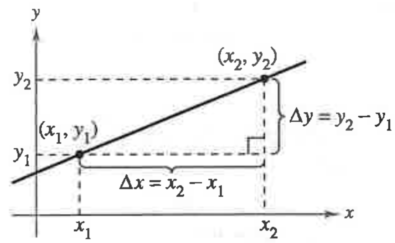
$$y = ax^2 + bx + c$$

has only one x-intercept.

P.2 Linear Models and Rates of Change

- **Find the slope of a line passing through two points.**
- **Write the equation of a line with a given point and slope.**
- **Interpret slope as a ratio or as a rate in a real-life application.**
- **Sketch the graph of a linear equation in slope-intercept form.**
- **Write equations of lines that are parallel or perpendicular to a given line.**

The Slope of a Line



$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x
Figure P.12

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure P.12. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (The symbol Δ is the uppercase Greek letter delta, and the symbols Δy and Δx are read "delta y" and "delta x.")

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

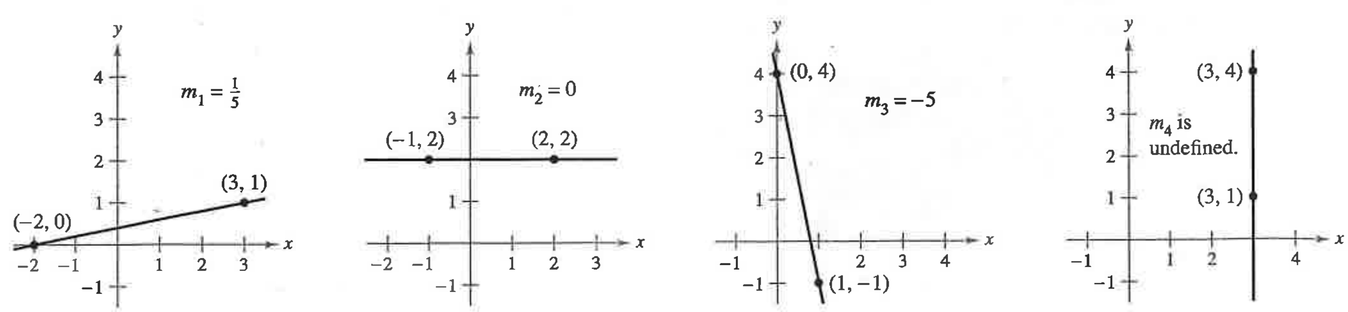
Slope is not defined for vertical lines.

When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point.

Figure P.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined" slope. In general, the greater the absolute value of the slope of a line, the steeper the line. For instance, in Figure P.13, the line with a slope of -5 is steeper than the line with a slope of $\frac{1}{5}$.



If m is positive, then the line rises from left to right.

If m is zero, then the line is horizontal.

If m is negative, then the line falls from left to right.

If m is undefined, then the line is vertical.

Figure P.13

Exploration

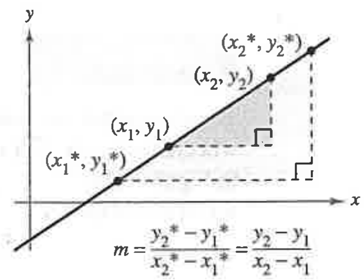
Investigating Equations of Lines Use a graphing utility to graph each of the linear equations. Which point is common to all seven lines? Which value in the equation determines the slope of each line?

a. $y - 4 = -2(x + 1)$
b. $y - 4 = -1(x + 1)$
c. $y - 4 = -\frac{1}{2}(x + 1)$
d. $y - 4 = 0(x + 1)$
e. $y - 4 = \frac{1}{2}(x + 1)$
f. $y - 4 = 1(x + 1)$
g. $y - 4 = 2(x + 1)$

Use your results to write an equation of a line passing through $(-1, 4)$ with a slope of m.

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure P.14. (Recall that the ratios of corresponding sides of similar triangles are equal.)



Any two points on a nonvertical line can be used to determine its slope.
Figure P.14

If (x_1, y_1) is a point on a nonvertical line that has a slope of m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

which is the **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

REMARK Remember that only nonvertical lines have a slope. Consequently, vertical lines cannot be written in point-slope form. For instance, the equation of the vertical line passing through the point $(1, -2)$ is $x = 1$.

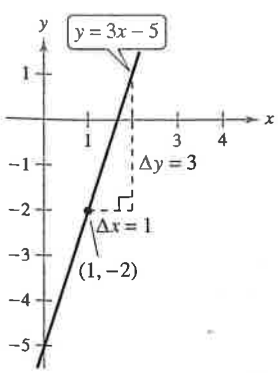
EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$. Then sketch the line.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute -2 for y_1, 1 for x_1, and 3 for m.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Solve for y.

To sketch the line, first plot the point $(1, -2)$. Then, because the slope is $m = 3$, you can locate a second point on the line by moving one unit to the right and three units upward, as shown in Figure P.15. ■



The line with a slope of 3 passing through the point $(1, -2)$
Figure P.15

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x- and y-axes have the same unit of measure, then the slope has no units and is a **ratio.** If the x- and y-axes have different units of measure, then the slope is a rate or **rate of change.** In your study of calculus, you will encounter applications involving both interpretations of slope.

EXAMPLE 2 Using Slope as a Ratio

The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business installs a wheelchair ramp that rises to a height of 22 inches over a length of 24 feet, as shown in Figure P.16. Is the ramp steeper than recommended? *(Source: ADA Standards for Accessible Design)*

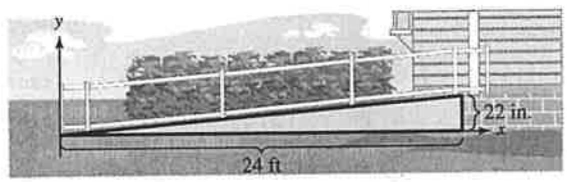


Figure P.16

Solution The length of the ramp is 24 feet or $12(24) = 288$ inches. The slope of the ramp is the ratio of its height (the rise) to its length (the run).

$$\text{Slope of ramp} = \frac{\text{rise}}{\text{run}} = \frac{22 \text{ in.}}{288 \text{ in.}} \approx 0.076$$

Because the slope of the ramp is less than $\frac{1}{12} \approx 0.083$, the ramp is not steeper than recommended. Note that the slope is a ratio and has no units.

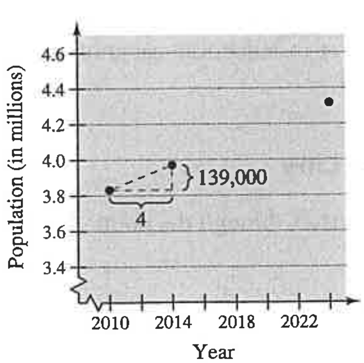
EXAMPLE 3 Using Slope as a Rate of Change

The population of Oregon was about 3,831,000 in 2010 and about 3,970,000 in 2014. Find the average rate of change of the population over this four-year period. What will the population of Oregon be in 2024? *(Source: U.S. Census Bureau)*

Solution Over this four-year period, the average rate of change of the population of Oregon was

$$\text{Rate of change} = \frac{\text{change in population}}{\text{change in years}} = \frac{3{,}970{,}000 - 3{,}831{,}000}{2014 - 2010} = 34{,}750 \text{ people per year.}$$

Assuming that Oregon's population continues to increase at this same rate for the next 10 years, it will have a 2024 population of about 4,318,000. (See Figure P.17.) ■



Population of Oregon
Figure P.17

The rate of change found in Example 3 is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is [2010, 2014]. In Chapter 2, you will study another type of rate of change called an *instantaneous rate of change.*

Graphing Linear Models

Many problems in coordinate geometry can be classified into two basic categories.

1. Given a graph (or parts of it), find its equation.
2. Given an equation, sketch its graph.

For lines, problems in the first category can be solved by using the point-slope form. The point-slope form, however, is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept** form of the equation of a line.

> **The Slope-Intercept Form of the Equation of a Line**
>
> The graph of the linear equation
>
> $y = mx + b$ Slope-intercept form
>
> is a line whose slope is m and whose y-intercept is $(0, b)$.

EXAMPLE 4 **Sketching Lines in the Plane**

Sketch the graph of each equation.

a. $y = 2x + 1$

b. $y = 2$

c. $3y + x - 6 = 0$

Solution

a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises two units for each unit it moves to the right, as shown in Figure P.18(a).

b. By writing the equation $y = 2$ in slope-intercept form

$$y = (0)x + 2$$

you can see that the slope is $m = 0$ and the y-intercept is $(0, 2)$. Because the slope is zero, you know that the line is horizontal, as shown in Figure P.18(b).

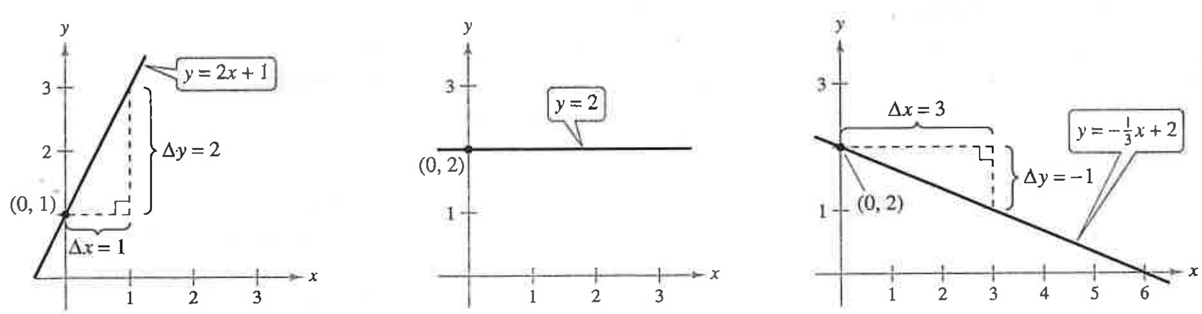
c. Begin by writing the equation in slope-intercept form.

$3y + x - 6 = 0$ Write original equation.

$3y = -x + 6$ Isolate y-term on the left.

$y = -\frac{1}{3}x + 2$ Slope-intercept form

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls one unit for every three units it moves to the right, as shown in Figure P.18(c).



(a) $m = 2$; line rises (b) $m = 0$; line is horizontal (c) $m = -\frac{1}{3}$; line falls

Figure P.18

Because the slope of a vertical line is not defined, its equation cannot be written in slope-intercept form. However, the equation of any line can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where A and B are not *both* zero. For instance, the vertical line

$$x = a \qquad \text{Vertical line}$$

can be represented by the general form

$$x - a = 0. \qquad \text{General form}$$

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

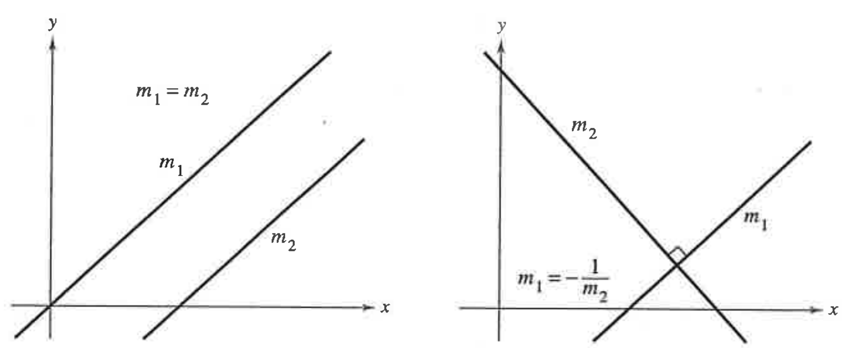
Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure P.19. Specifically, nonvertical lines with the same slope are parallel, and nonvertical lines whose slopes are negative reciprocals are perpendicular.

REMARK In mathematics, the phrase "if and only if" is a way of stating two implications in one statement. For instance, the first statement at the right could be rewritten as the following two implications.

a. If two distinct nonvertical lines are parallel, then their slopes are equal.

b. If two distinct nonvertical lines have equal slopes, then they are parallel.



Parallel lines Perpendicular lines

Figure P.19

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

$$m_1 = m_2. \qquad \text{Parallel} \Longleftrightarrow \text{Slopes are equal.}$$

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

$$m_1 = -\frac{1}{m_2}. \qquad \text{Perpendicular} \Longleftrightarrow \text{Slopes are negative reciprocals.}$$

EXAMPLE 5 Finding Parallel and Perpendicular Lines

See LarsonCalculus.com for an interactive version of this type of example.

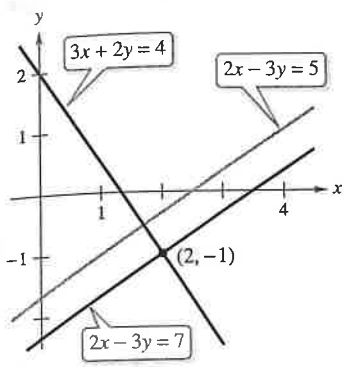
Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

Solution Begin by writing the linear equation $2x - 3y = 5$ in slope-intercept form.

$2x - 3y = 5$ — Write original equation.

$y = \frac{2}{3}x - \frac{5}{3}$ — Slope-intercept form

So, the given line has a slope of $m = \frac{2}{3}$. (See Figure P.20.)



Lines parallel and perpendicular to $2x - 3y = 5$

Figure P.20

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

$y - y_1 = m(x - x_1)$ — Point-slope form

$y - (-1) = \frac{2}{3}(x - 2)$ — Substitute.

$3(y + 1) = 2(x - 2)$ — Simplify.

$3y + 3 = 2x - 4$ — Distributive Property

$2x - 3y - 7 = 0$ — General form

Note the similarity to the equation of the given line, $2x - 3y = 5$.

b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$.

$y - y_1 = m(x - x_1)$ — Point-slope form

$y - (-1) = -\frac{3}{2}(x - 2)$ — Substitute.

$2(y + 1) = -3(x - 2)$ — Simplify.

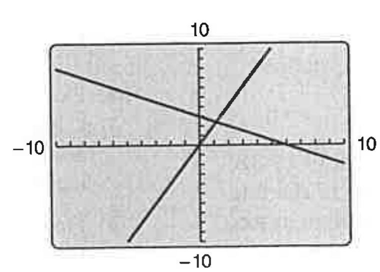
$2y + 2 = -3x + 6$ — Distributive Property

$3x + 2y - 4 = 0$ — General form ■

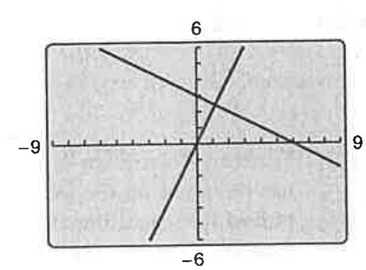
TECHNOLOGY PITFALL The slope of a line will appear distorted if you use different tick-mark spacing on the x- and y-axes. For instance, the graphing utility screens in Figures P.21(a) and P.21(b) both show the lines

$$y = 2x \quad \text{and} \quad y = -\tfrac{1}{2}x + 3.$$

Because these lines have slopes that are negative reciprocals, they must be perpendicular. In Figure P.21(a), however, the lines do not appear to be perpendicular because the tick-mark spacing on the x-axis is not the same as that on the y-axis. In Figure P.21(b), the lines appear perpendicular because the tick-mark spacing on the x-axis is the same as on the y-axis. This type of viewing window is said to have a *square setting*.



(a) Tick-mark spacing on the x-axis is not the same as tick-mark spacing on the y-axis.



(b) Tick-mark spacing on the x-axis is the same as tick-mark spacing on the y-axis.

Figure P.21

P.2 Exercises

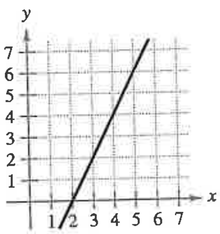
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Slope-Intercept Form** In the form $y = mx + b$, what does m represent? What does b represent?
2. **Perpendicular Lines** Is it possible for two lines with positive slopes to be perpendicular? Why or why not?

Estimating Slope **In Exercises 3–6, estimate the slope of the line from its graph. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

3.



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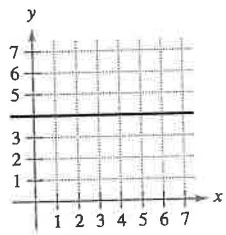

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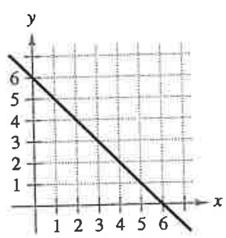

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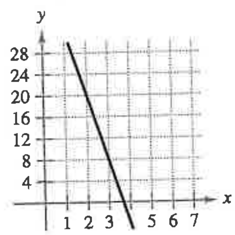


Finding the Slope of a Line **In Exercises 7–12, plot the pair of points and find the slope of the line passing through them.**

7. $(3, -4), (5, 2)$
8. $(0, 0), (-2, 3)$
9. $(4, 6), (4, 1)$
10. $(3, -5), (5, -5)$
11. $\left(-\frac{1}{2}, \frac{2}{3}\right), \left(-\frac{3}{4}, \frac{1}{6}\right)$
12. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

Sketching Lines **In Exercises 13 and 14, sketch the lines through the point with the indicated slopes. Make the sketches on the same set of coordinate axes.**

	Point	Slopes			
13.	$(3, 4)$	(a) 1	(b) -2	(c) $-\frac{3}{2}$	(d) Undefined
14.	$(-2, 5)$	(a) 3	(b) -3	(c) $\frac{1}{3}$	(d) 0

Finding Points on a Line **In Exercises 15–18, use the point on the line and the slope of the line to find three additional points that the line passes through. (There is more than one correct answer.)**

	Point	Slope		Point	Slope
15.	$(6, 2)$	$m = 0$	16.	$(-4, 3)$	m is undefined.
17.	$(1, 7)$	$m = -3$	18.	$(-2, -2)$	$m = 2$

Finding an Equation of a Line **In Exercises 19–24, find an equation of the line that passes through the point and has the indicated slope. Then sketch the line.**

	Point	Slope
19.	$(0, 3)$	$m = \frac{3}{4}$
20.	$(-5, -2)$	$m = \frac{6}{5}$
21.	$(1, 2)$	m is undefined.
22.	$(0, 4)$	$m = 0$
23.	$(3, -2)$	$m = 3$
24.	$(-2, 4)$	$m = -\frac{3}{5}$

25. **Road Grade** You are driving on a road that has a 6% uphill grade. This means that the slope of the road is $\frac{6}{100}$. Approximate the amount of vertical change in your position when you drive 200 feet.

26. **Conveyor Design**

A moving conveyor is built to rise 1 meter for each 3 meters of horizontal change.

(a) Find the slope of the conveyor.

(b) Suppose the conveyor runs between two floors in a factory. Find the length of the conveyor when the vertical distance between floors is 10 feet.

27. **Modeling Data** The table shows the populations y (in millions) of the United States for 2009 through 2014. The variable t represents the time in years, with $t = 9$ corresponding to 2009. *(Source: U.S. Census Bureau)*

t	9	10	11	12	13	14
y	307.0	309.3	311.7	314.1	316.5	318.9

(a) Plot the data by hand and connect adjacent points with a line segment. Use the slope of each line segment to determine the year when the population increased least rapidly.

(b) Find the average rate of change of the population of the United States from 2009 through 2014.

(c) Use the average rate of change of the population to predict the population of the United States in 2025.

28. **Biodiesel Production** The table shows the biodiesel productions y (in thousands of barrels per day) for the United States for 2007 through 2012. The variable t represents the time in years, with $t = 7$ corresponding to 2007. *(Source: U.S. Energy Information Administration)*

t	7	8	9	10	11	12
y	32	44	34	22	63	64

(a) Plot the data by hand and connect adjacent points with a line segment. Use the slope of each line segment to determine the year when biodiesel production increased most rapidly.

(b) Find the average rate of change of biodiesel production for the United States from 2007 through 2012.

(c) Should the average rate of change be used to predict future biodiesel production? Explain.

Finding the Slope and y-Intercept **In Exercises 29–34, find the slope and the y-intercept (if possible) of the line.**

29. $y = 4x - 3$
30. $-x + y = 1$
31. $5x + y = 20$
32. $6x - 5y = 15$
33. $x = 4$
34. $y = -1$

Sketching a Line in the Plane **In Exercises 35–42, sketch the graph of the equation.**

35. $y = -3$
36. $x = 4$
37. $y = -2x + 1$
38. $y = \frac{1}{3}x - 1$
39. $y - 2 = \frac{3}{2}(x - 1)$
40. $y - 1 = 3(x + 4)$
41. $3x - 3y + 1 = 0$
42. $x + 2y + 6 = 0$

Finding an Equation of a Line **In Exercises 43–50, find an equation of the line that passes through the points. Then sketch the line.**

43. $(4, 3), (0, -5)$
44. $(-2, -2), (1, 7)$
45. $(2, 8), (5, 0)$
46. $(-3, 6), (1, 2)$
47. $(6, 3), (6, 8)$
48. $(1, -2), (3, -2)$
49. $(3, 1), (5, 1)$
50. $(2, 5), (2, 7)$

51. **Writing an Equation** Write an equation for the line that passes through the points $(0, b)$ and $(3, 1)$.

52. **Using Intercepts** Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

Writing an Equation in General Form **In Exercises 53–56, use the result of Exercise 52 to write an equation of the line with the given characteristics in general form.**

53. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

54. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

55. Point on line: $(9, -2)$
x-intercept: $(2a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

56. Point on line: $\left(-\frac{2}{3}, -2\right)$
x-intercept: $(a, 0)$
y-intercept: $(0, -a)$
$(a \neq 0)$

Finding Parallel and Perpendicular Lines **In Exercises 57–62, write the general forms of the equations of the lines that pass through the point and are (a) parallel to the given line and (b) perpendicular to the given line.**

	Point	Line
57.	$(-7, -2)$	$x = 1$
58.	$(-1, 0)$	$y = -3$
59.	$(-3, 2)$	$x + y = 7$
60.	$(2, 5)$	$x - y = -2$
61.	$\left(\frac{3}{4}, \frac{7}{8}\right)$	$5x - 3y = 0$
62.	$\left(\frac{5}{6}, -\frac{1}{2}\right)$	$7x + 4y = 8$

Rate of Change **In Exercises 63 and 64, you are given the dollar value of a product in 2016 *and* the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2010.)**

	2016 Value	Rate
63.	\$1850	\$250 increase per year
64.	\$17,200	\$1600 decrease per year

Collinear Points **In Exercises 65 and 66, determine whether the points are collinear. (Three points are *collinear* if they lie on the same line.)**

65. $(-2, 1), (-1, 0), (2, -2)$

66. $(0, 4), (7, -6), (-5, 11)$

EXPLORING CONCEPTS

67. **Square** Show that the points $(-1, 0)$, $(3, 0)$, $(1, 2)$, and $(1, -2)$ are vertices of a square.

68. **Analyzing a Line** A line is represented by the equation $ax + by = 4$.

(a) When is the line parallel to the x-axis?

(b) When is the line parallel to the y-axis?

(c) Give values for a and b such that the line has a slope of $\frac{5}{8}$.

(d) Give values for a and b such that the line is perpendicular to $y = \frac{2}{5}x + 3$.

(e) Give values for a and b such that the line coincides with the graph of $5x + 6y = 8$.

69. **Tangent Line** Find an equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point $(5, 12)$.

70. **Tangent Line** Find an equation of the line tangent to the circle $(x - 1)^2 + (y - 1)^2 = 25$ at the point $(4, -3)$.

71. Finding Points of Intersection Find the coordinates of the point of intersection of the given segments. Explain your reasoning.

(a) Perpendicular bisectors (b) Medians

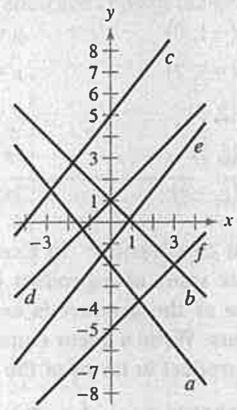
72. HOW DO YOU SEE IT? Several lines are shown in the figure below. (The lines are labeled a–f.)

(a) Which lines have a positive slope?

(b) Which lines have a negative slope?

(c) Which lines appear parallel?

(d) Which lines appear perpendicular?



73. Temperature Conversion Find a linear equation that expresses the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0°C (32°F) and boils at 100°C (212°F). Use the equation to convert 72°F to degrees Celsius.

74. Choosing a Job As a salesperson, you receive a monthly salary of \$2000, plus a commission of 7% of sales. You are offered a new job at \$2300 per month, plus a commission of 5% of sales.

(a) Write linear equations for your monthly wage W in terms of your monthly sales s for your current job and your job offer.

(b) Use a graphing utility to graph each equation and find the point of intersection. What does it signify?

(c) You think you can sell \$20,000 worth of a product per month. Should you change jobs? Explain.

75. Apartment Rental A real estate office manages an apartment complex with 50 units. When the rent is \$780 per month, all 50 units are occupied. However, when the rent is \$825, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear. (*Note:* The term *demand* refers to the number of occupied units.)

(a) Write a linear equation giving the demand x in terms of the rent p.

(b) *Linear extrapolation* Use a graphing utility to graph the demand equation and use the *trace* feature to predict the number of units occupied when the rent is raised to \$855.

(c) *Linear interpolation* Predict the number of units occupied when the rent is lowered to \$795. Verify graphically.

76. Modeling Data An instructor gives regular 20-point quizzes and 100-point exams in a mathematics course. Average scores for six students, given as ordered pairs (x, y), where x is the average quiz score and y is the average exam score, are (18, 87), (10, 55), (19, 96), (16, 79), (13, 76), and (15, 82).

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the points and graph the regression line in the same viewing window.

(c) Use the regression line to predict the average exam score for a student with an average quiz score of 17.

(d) Interpret the meaning of the slope of the regression line.

(e) The instructor adds 4 points to the average exam score of everyone in the class. Describe the changes in the positions of the plotted points and the change in the equation of the line.

77. Distance Show that the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

78. Distance Write the distance d between the point (3, 1) and the line $y = mx + 4$ in terms of m. Use a graphing utility to graph the equation. When is the distance 0? Explain the result geometrically.

Distance **In Exercises 79 and 80, use the result of Exercise 77 to find the distance between the point and line.**

79. Point: $(-2, 1)$
Line: $x - y - 2 = 0$

80. Point: $(2, 3)$
Line: $4x + 3y = 10$

81. Proof Prove that the diagonals of a rhombus intersect at right angles. (A rhombus is a quadrilateral with sides of equal lengths.)

82. Proof Prove that the figure formed by connecting consecutive midpoints of the sides of any quadrilateral is a parallelogram.

83. Proof Prove that if the points (x_1, y_1) and (x_2, y_2) lie on the same line as (x_1^*, y_1^*) and (x_2^*, y_2^*), then

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Assume $x_1 \neq x_2$ and $x_1^* \neq x_2^*$.

84. Proof Prove that if the slopes of two nonvertical lines are negative reciprocals of each other, then the lines are perpendicular.

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. The lines represented by

$$ax + by = c_1 \quad \text{and} \quad bx - ay = c_2$$

are perpendicular. Assume $a \neq 0$ and $b \neq 0$.

86. If a line contains points in both the first and third quadrants, then its slope must be positive.

P.3 Functions and Their Graphs

- **Use function notation to represent and evaluate a function.**
- **Find the domain and range of a function.**
- **Sketch the graph of a function.**
- **Identify different types of transformations of functions.**
- **Classify functions and recognize combinations of functions.**

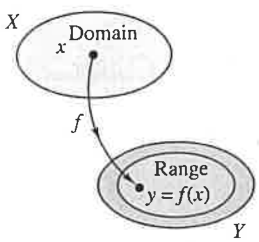
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$$A = \pi r^2 \qquad \text{A is a function of r.}$$

In this case, r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable
Figure P.22

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X. (See Figure P.22.)

Functions can be specified in a variety of ways. In this text, however, you will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$$x^2 + 2y = 1 \qquad \text{Equation in implicit form}$$

defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \tfrac{1}{2}(1 - x^2) \qquad \text{Equation in explicit form}$$

Using f as the name of the function, you can write this equation as

$$f(x) = \tfrac{1}{2}(1 - x^2). \qquad \text{Function notation}$$

The original equation

$$x^2 + 2y = 1$$

implicitly defines y as a function of x. When you solve the equation for y, you are writing the equation in **explicit** form.

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The symbol $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

FUNCTION NOTATION

The word *function* was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word "function" to describe any expression made up of a variable and some constants. He introduced the notation $y = f(x)$. (To read more about Euler, see the biography on the next page.)

LEONHARD EULER (1707–1783)

In addition to making major contributions to almost every branch of mathematics, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics, and magnetism.

See LarsonCalculus.com to read more of this biography.

In an equation that defines a function of x, the role of the variable x is simply that of a placeholder. For instance, the function

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\square) = 2(\square)^2 - 4(\square) + 1$$

where rectangles are used instead of x. To evaluate $f(-2)$, replace each rectangle with -2.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 && \text{Substitute } -2 \text{ for } x. \\ &= 2(4) + 8 + 1 && \text{Simplify.} \\ &= 17 && \text{Simplify.} \end{aligned}$$

Although f is often used as a convenient function name with x as the independent variable, you can use other symbols. For instance, these three equations all define the same function.

$$\begin{aligned} f(x) &= x^2 - 4x + 7 && \text{Function name is } f\text{, independent variable is } x. \\ f(t) &= t^2 - 4t + 7 && \text{Function name is } f\text{, independent variable is } t. \\ g(s) &= s^2 - 4s + 7 && \text{Function name is } g\text{, independent variable is } s. \end{aligned}$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate each expression.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Solution

a. $$\begin{aligned} f(3a) &= (3a)^2 + 7 && \text{Substitute } 3a \text{ for } x. \\ &= 9a^2 + 7 && \text{Simplify.} \end{aligned}$$

b. $$\begin{aligned} f(b - 1) &= (b - 1)^2 + 7 && \text{Substitute } b - 1 \text{ for } x. \\ &= b^2 - 2b + 1 + 7 && \text{Expand binomial.} \\ &= b^2 - 2b + 8 && \text{Simplify.} \end{aligned}$$

c. $$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} \\ &= 2x + \Delta x, \quad \Delta x \neq 0 \end{aligned}$$

REMARK The expression in Example 1(c) is called a *difference quotient* and has a special significance in calculus. You will learn more about this in Chapter 2.

In calculus, it is important to specify the domain of a function or expression clearly. For instance, in Example 1(c), the two expressions

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad 2x + \Delta x, \quad \Delta x \neq 0$$

are equivalent because $\Delta x = 0$ is excluded from the domain of each expression. Without a stated domain restriction, the two expressions would not be equivalent.

North Wind Picture Archives / Alamy Stock Photo

The Domain and Range of a Function

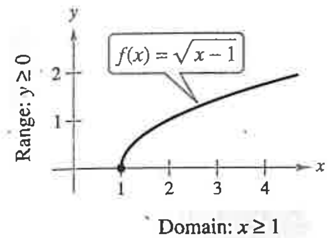
The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The **implied domain** is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

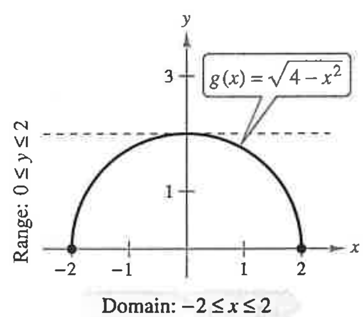
has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x: x \ne \pm 2\}$.



(a) The domain of f is $[1, \infty)$, and the range is $[0, \infty)$.



(b) The domain of g is $[-2, 2]$, and the range is $[0, 2]$.

Figure P.23

EXAMPLE 2 Finding the Domain and Range of a Function

Find the domain and range of each function.

a. $f(x) = \sqrt{x - 1}$ **b.** $g(x) = \sqrt{4 - x^2}$

Solution

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. So, the range is the interval $[0, \infty)$, as shown in Figure P.23(a).

b. The domain of the function

$$g(x) = \sqrt{4 - x^2}$$

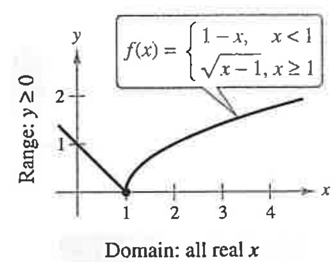
is the set of all values for which $4 - x^2 \ge 0$, or $x^2 \le 4$. So, the domain of g is the interval $[-2, 2]$. To find the range, observe that $g(x) = \sqrt{4 - x^2}$ is never negative and is at most 2. So, the range is the interval $[0, 2]$, as shown in Figure P.23(b). Note that the graph of g is a *semicircle* of radius 2.

EXAMPLE 3 A Function Defined by More than One Equation

For the piecewise-defined function

$$f(x) = \begin{cases} 1 - x, & x < 1 \\ \sqrt{x - 1}, & x \ge 1 \end{cases}$$

f is defined for $x < 1$ and $x \ge 1$. So, the domain is the set of all real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2(a). For $x < 1$, the values of $1 - x$ are positive. So, the range of the function is the interval $[0, \infty)$. (See Figure P.24.)



The domain of f is $(-\infty, \infty)$, and the range is $[0, \infty)$.

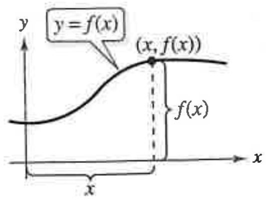
Figure P.24

THE SQUARE ROOT SYMBOL

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. The symbol was chosen because it resembled a lowercase r, to stand for the Latin word *radix*, meaning root.

A function from X to Y is **one-to-one** when to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function in Example 2(a) is one-to-one, whereas the functions in Examples 2(b) and 3 are not one-to-one. A function from X to Y is **onto** when its range consists of all of Y.

The Graph of a Function



The graph of a function
Figure P.25

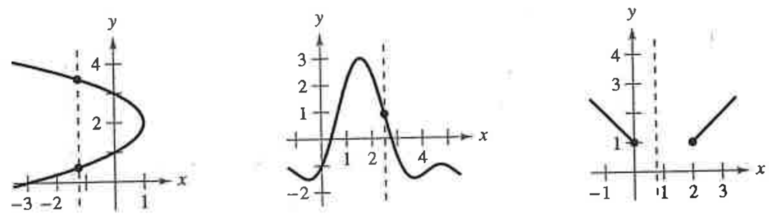
The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure P.25, note that

$x =$ the directed distance from the y-axis

and

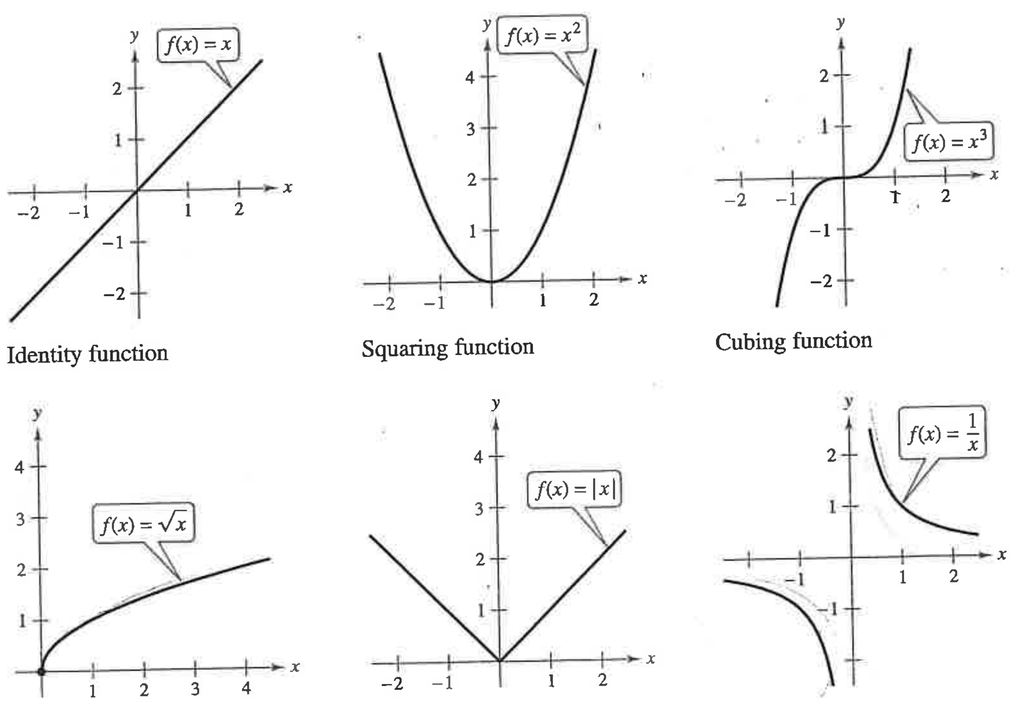
$f(x) =$ the directed distance from the x-axis.

A vertical line can intersect the graph of a function of x at most *once*. This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x. That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point. For example, in Figure P.26(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice, whereas in Figures P.26(b) and (c), the graphs do define y as a function of x.



(a) Not a function of x **(b)** A function of x **(c)** A function of x
Figure P.26

Figure P.27 shows the graphs of six basic functions. You should be able to recognize these graphs. (The graphs of the six basic trigonometric functions are shown in Section P.4.)



Identity function Squaring function Cubing function

Square root function Absolute value function Rational function

The graphs of six basic functions
Figure P.27

Transformations of Functions

Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure P.28.

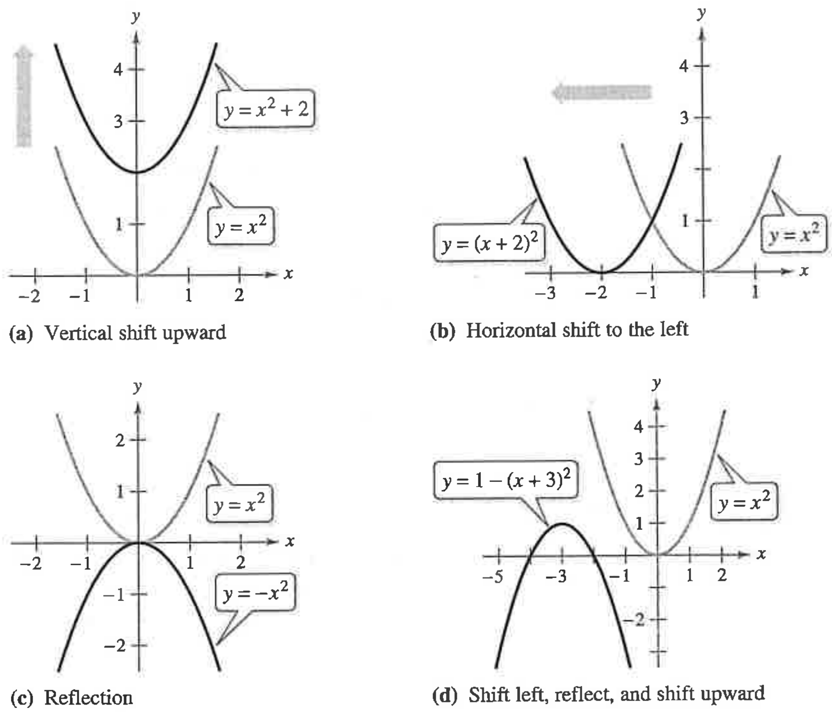


Figure P.28

Each of the graphs in Figure P.28 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane. For instance, using

$f(x) = x^2$ Original function

as the original function, the transformations shown in Figure P.28 can be represented by these equations.

a. $y = f(x) + 2$ Vertical shift up two units

b. $y = f(x + 2)$ Horizontal shift to the left two units

c. $y = -f(x)$ Reflection about the x-axis

d. $y = -f(x + 3) + 1$ Shift left three units, reflect about the x-axis, and shift up one unit

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Classifications and Combinations of Functions

FOR FURTHER INFORMATION For more on the history of the concept of a function, see the article "Evolution of the Function Concept: A Brief Survey" by Israel Kleiner in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler, who introduced the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.** Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

You will review the trigonometric functions in the next section. The other nonalgebraic functions, such as the inverse trigonometric functions and the exponential and logarithmic functions, are introduced in Chapter 5.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

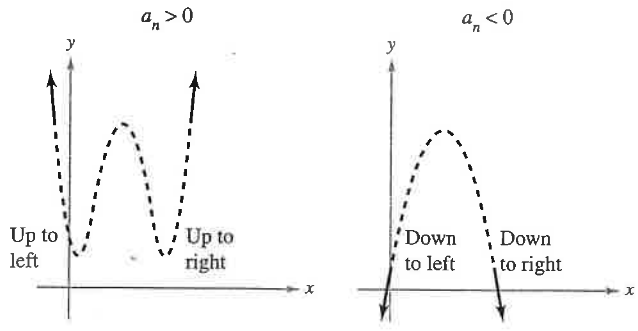
where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the **degree** of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, these simpler forms are often used. (Note that $a \neq 0$.)

Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

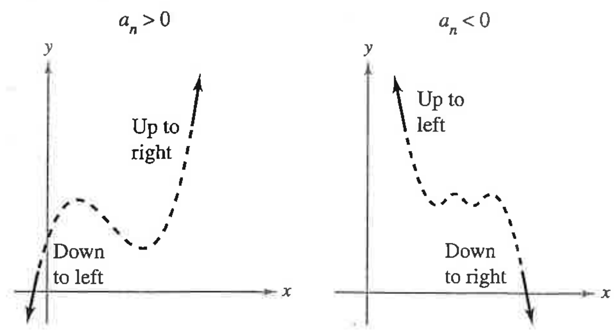
Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure P.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



Graphs of polynomial functions of even degree



Graphs of polynomial functions of odd degree

The Leading Coefficient Test for polynomial functions
Figure P.29

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational when it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

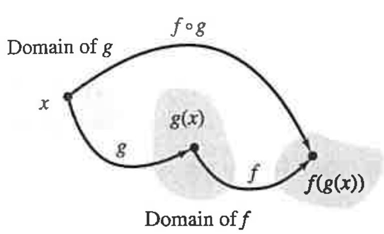
where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example, $f(x) = \sqrt{x+1}$ is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions (see Section P.4) are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$$(f+g)(x) = f(x) + g(x) = (2x-3) + (x^2+1) \qquad \text{Sum}$$
$$(f-g)(x) = f(x) - g(x) = (2x-3) - (x^2+1) \qquad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = (2x-3)(x^2+1) \qquad \text{Product}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x-3}{x^2+1} \qquad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**



The domain of the composite function $f \circ g$
Figure P.30

> **Definition of Composite Function**
>
> Let f and g be functions. The function $(f \circ g)(x) = f(g(x))$ is the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure P.30).

The composite of f with g is generally not the same as the composite of g with f. This is shown in the next example.

EXAMPLE 4 Finding Composite Functions

See LarsonCalculus.com for an interactive version of this type of example.

For $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, find each composite function.

a. $f \circ g$ **b.** $g \circ f$

Solution

a. $(f \circ g)(x) = f(g(x))$ — Definition of $f \circ g$

$= f(x^2 + 1)$ — Substitute $x^2 + 1$ for $g(x)$.

$= 2(x^2 + 1) - 3$ — Definition of $f(x)$

$= 2x^2 - 1$ — Simplify.

b. $(g \circ f)x = g(f(x))$ — Definition of $g \circ f$

$= g(2x - 3)$ — Substitute $2x - 3$ for $f(x)$.

$= (2x - 3)^2 + 1$ — Definition of $g(x)$

$= 4x^2 - 12x + 10$ — Simplify.

Note that $(f \circ g)(x) \neq (g \circ f)(x)$. ■

In Section P.1, an x-intercept of a graph was defined to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, then the number a is a **zero** of f. In other words, *the zeros of a function f are the solutions of the equation* $f(x) = 0$. For example, the function

$$f(x) = x - 4$$

has a zero at $x = 4$ because $f(4) = 0$.

In Section P.1, you also studied different types of symmetry. In the terminology of functions, a function $y = f(x)$ is **even** when its graph is symmetric with respect to the y-axis, and is **odd** when its graph is symmetric with respect to the origin. The symmetry tests in Section P.1 yield the following test for even and odd functions.

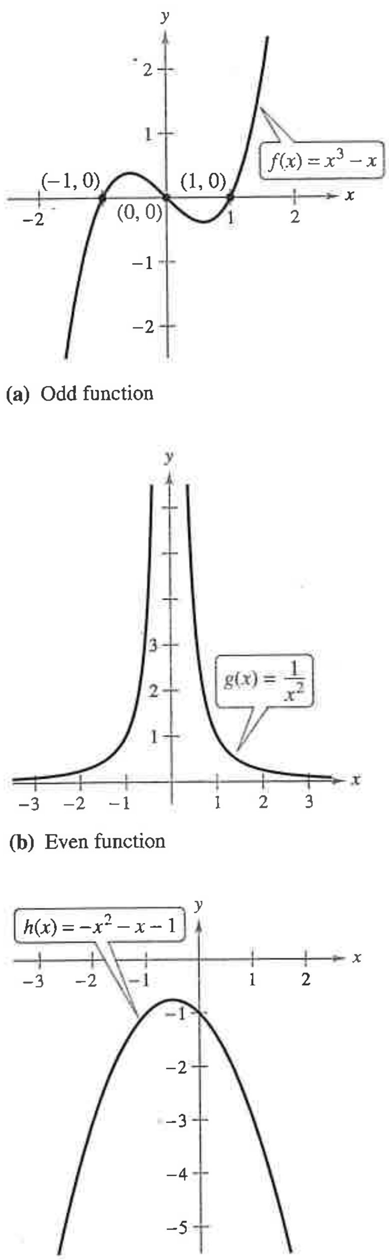
Test for Even and Odd Functions

The function $y = f(x)$ is **even** when

$$f(-x) = f(x).$$

The function $y = f(x)$ is **odd** when

$$f(-x) = -f(x).$$



(a) Odd function

(b) Even function

(c) Neither even nor odd

Figure P.31

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = \dfrac{1}{x^2}$ **c.** $h(x) = -x^2 - x - 1$

Solution

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are

$x^3 - x = 0$	Let $f(x) = 0$.
$x(x^2 - 1) = 0$	Factor.
$x(x - 1)(x + 1) = 0$	Factor.
$x = 0, 1, -1.$	Zeros of f

See Figure P.31(a).

b. This function is even because

$$g(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = g(x).$$

This function does not have zeros because $1/x^2$ is positive for all x in the domain, as shown in Figure P.31(b).

c. Substituting $-x$ for x produces

$$h(-x) = -(-x)^2 - (-x) - 1 = -x^2 + x - 1.$$

Because $h(x) = -x^2 - x - 1$ and $-h(x) = x^2 + x + 1$, you can conclude that

$h(-x) \neq h(x)$ Function is *not* even.

and

$h(-x) \neq -h(x).$ Function is *not* odd.

So, the function is neither even nor odd. This function does not have zeros because $-x^2 - x - 1$ is negative for all x, as shown in Figure P.31(c).

P.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Writing** Describe how a relation and a function are different.
2. **Domain and Range** In your own words, explain the meanings of *domain* and *range*.
3. **Transformations** What are the three basic types of function transformations?
4. **Right and Left Behavior** Describe the four cases of the Leading Coefficient Test.

Evaluating a Function **In Exercises 5–12, evaluate the function at the given value(s) of the independent variable. Simplify the results.**

5. $f(x) = 3x - 2$
   (a) $f(0)$ (b) $f(5)$ (c) $f(b)$ (d) $f(x - 1)$
6. $f(x) = 7x - 4$
   (a) $f(0)$ (b) $f(-3)$ (c) $f(b)$ (d) $f(x + 2)$
7. $f(x) = \sqrt{x^2 + 4}$
   (a) $f(-2)$ (b) $f(3)$ (c) $f(2)$ (d) $f(x + bx)$
8. $f(x) = \sqrt{x + 5}$
   (a) $f(-4)$ (b) $f(11)$ (c) $f(4)$ (d) $f(x + \Delta x)$
9. $g(x) = 5 - x^2$
   (a) $g(0)$ (b) $g(\sqrt{5})$ (c) $g(-2)$ (d) $g(t - 1)$
10. $g(x) = x^2(x - 4)$
    (a) $g(4)$ (b) $g(\frac{3}{2})$ (c) $g(c)$ (d) $g(t + 4)$
11. $f(x) = x^3$
    $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$
12. $f(x) = 3x - 1$
    $\dfrac{f(x) - f(1)}{x - 1}$

Finding the Domain and Range of a Function **In Exercises 13–22, find the domain and range of the function.**

13. $f(x) = 4x^2$
14. $g(x) = x^2 - 5$
15. $f(x) = x^3$
16. $h(x) = 4 - x^2$
17. $g(x) = \sqrt{6x}$
18. $h(x) = -\sqrt{x + 3}$
19. $f(x) = \sqrt{16 - x^2}$
20. $f(x) = |x - 3|$
21. $f(x) = \dfrac{3}{x}$
22. $f(x) = \dfrac{x - 2}{x + 4}$

Finding the Domain of a Function **In Exercises 23–26, find the domain of the function.**

23. $f(x) = \sqrt{x} + \sqrt{1 - x}$
24. $f(x) = \sqrt{x^2 - 3x + 2}$
25. $f(x) = \dfrac{1}{|x + 3|}$
26. $g(x) = \dfrac{1}{|x^2 - 4|}$

Finding the Domain and Range of a Piecewise Function **In Exercises 27–30, evaluate the function at the given value(s) of the independent variable. Then find the domain and range.**

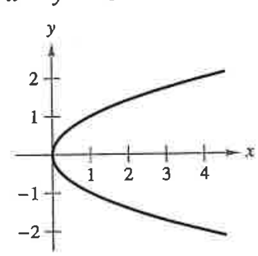
27. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
    (a) $f(-1)$ (b) $f(0)$ (c) $f(2)$ (d) $f(t^2 + 1)$
28. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$
    (a) $f(-2)$ (b) $f(0)$ (c) $f(1)$ (d) $f(s^2 + 2)$
29. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$
    (a) $f(-3)$ (b) $f(1)$ (c) $f(3)$ (d) $f(b^2 + 1)$
30. $f(x) = \begin{cases} \sqrt{x + 4}, & x \leq 5 \\ (x - 5)^2, & x > 5 \end{cases}$
    (a) $f(-3)$ (b) $f(0)$ (c) $f(5)$ (d) $f(10)$

Sketching a Graph of a Function **In Exercises 31–38, sketch a graph of the function and find its domain and range. Use a graphing utility to verify your graph.**

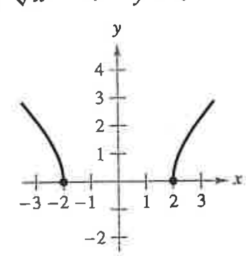
31. $f(x) = 4 - x$
32. $f(x) = x^2 + 5$
33. $g(x) = \dfrac{1}{x^2 + 2}$
34. $f(t) = \dfrac{2}{7 + t}$
35. $h(x) = \sqrt{x - 6}$
36. $f(x) = \frac{1}{4}x^3 + 3$
37. $f(x) = \sqrt{9 - x^2}$
38. $f(x) = x + \sqrt{4 - x^2}$

Using the Vertical Line Test **In Exercises 39–42, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

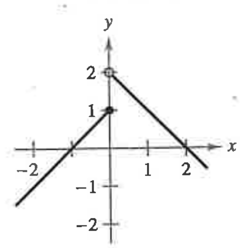
39. $x - y^2 = 0$



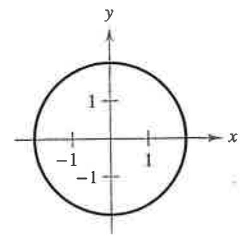
40. $\sqrt{x^2 - 4} - y = 0$



41. $y = \begin{cases} x + 1, & x \leq 0 \\ -x + 2, & x > 0 \end{cases}$



42. $x^2 + y^2 = 4$

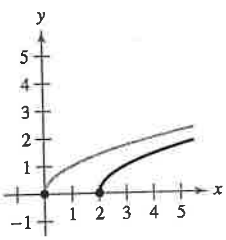


Deciding Whether an Equation Is a Function In Exercises 43–46, determine whether y is a function of x.

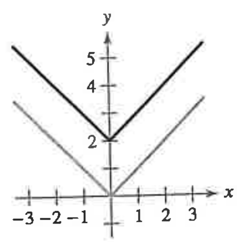
43. $x^2 + y^2 = 16$
44. $x^2 + y = 16$
45. $y^2 = x^2 - 1$
46. $x^2y - x^2 + 4y = 0$

Transformation of a Function In Exercises 47–50, the graph shows one of the six basic functions on page 22 and a transformation of the function. Describe the transformation. Then use your description to write an equation for the transformation.

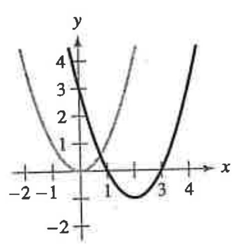
47.



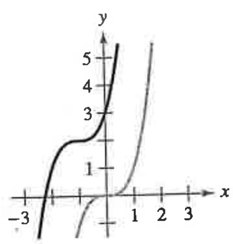
48.



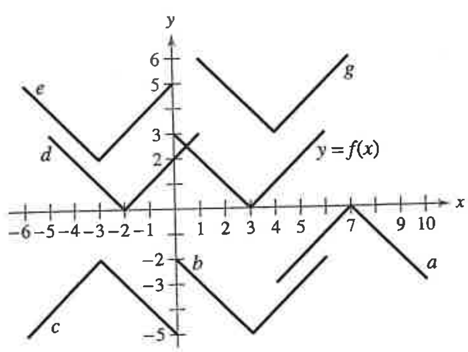
49.



50.



Matching In Exercises 51–56, use the graph of $y = f(x)$ to match the function with its graph.



51. $y = f(x + 5)$
52. $y = f(x) - 5$
53. $y = -f(-x) - 2$
54. $y = -f(x - 4)$
55. $y = f(x + 6) + 2$
56. $y = f(x - 1) + 3$

57. **Sketching Transformations** Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com.*

(a) $f(x + 3)$ (b) $f(x - 1)$
(c) $f(x) + 2$ (d) $f(x) - 4$
(e) $3f(x)$ (f) $\frac{1}{4}f(x)$
(g) $-f(x)$ (h) $-f(-x)$

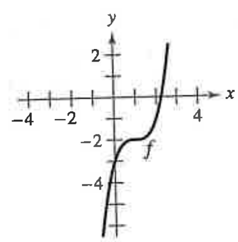


Figure for 57

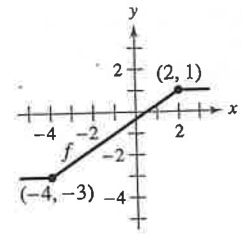


Figure for 58

58. **Sketching Transformations** Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com.*

(a) $f(x - 4)$ (b) $f(x + 2)$
(c) $f(x) + 4$ (d) $f(x) - 1$
(e) $2f(x)$ (f) $\frac{1}{2}f(x)$
(g) $f(-x)$ (h) $-f(x)$

Combinations of Functions In Exercises 59 and 60, find (a) $f(x) + g(x)$, (b) $f(x) - g(x)$, (c) $f(x) \cdot g(x)$, and (d) $f(x)/g(x)$.

59. $f(x) = 2x - 5$
$g(x) = 4 - 3x$

60. $f(x) = x^2 + 5x + 4$
$g(x) = x + 1$

61. **Evaluating Composite Functions** Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $g(f(0))$
(d) $f(g(-4))$ (e) $f(g(x))$ (f) $g(f(x))$

62. **Evaluating Composite Functions** Given $f(x) = 2x^3$ and $g(x) = 4x + 3$, evaluate each expression.

(a) $f(g(0))$ (b) $f\left(g\left(\frac{1}{2}\right)\right)$ (c) $g(f(0))$
(d) $g\left(f\left(-\frac{1}{4}\right)\right)$ (e) $f(g(x))$ (f) $g(f(x))$

Finding Composite Functions In Exercises 63–66, find the composite functions $f \circ g$ and $g \circ f$. Find the domain of each composite function. Are the two composite functions equal?

63. $f(x) = x^2$
$g(x) = \sqrt{x}$

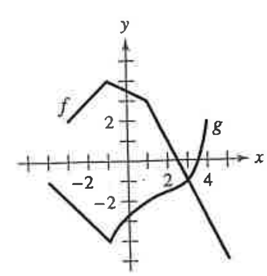
64. $f(x) = x^2 - 1$
$g(x) = -x$

65. $f(x) = \dfrac{3}{x}$
$g(x) = x^2 - 1$

66. $f(x) = \dfrac{1}{x}$
$g(x) = \sqrt{x + 2}$

67. **Evaluating Composite Functions** Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$
(b) $g(f(2))$
(c) $g(f(5))$
(d) $(f \circ g)(-3)$
(e) $(g \circ f)(-1)$
(f) $f(g(-1))$



68. **Ripples** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Think About It **In Exercises 69 and 70, $F(x) = f \circ g \circ h$. Identify functions for f, g, and h. (There are many correct answers.)**

69. $F(x) = \sqrt{2x - 2}$

70. $F(x) = \dfrac{1}{4x^6}$

Think About It **In Exercises 71 and 72, find the coordinates of a second point on the graph of a function f when the given point is on the graph and the function is (a) even and (b) odd.**

71. $\left(-\frac{3}{2}, 4\right)$

72. $(4, 9)$

73. **Even and Odd Functions** The graphs of f, g, and h are shown in the figure. Decide whether each function is even, odd, or neither.

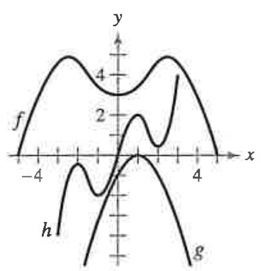


Figure for 73

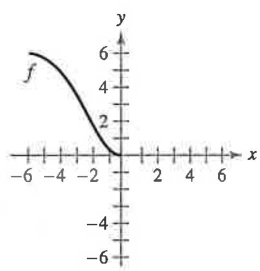


Figure for 74

74. **Even and Odd Functions** The domain of the function f shown in the figure is $-6 \le x \le 6$.

(a) Complete the graph of f given that f is even.

(b) Complete the graph of f given that f is odd.

Even and Odd Functions and Zeros of Functions **In Exercises 75–78, determine whether the function is even, odd, or neither. Then find the zeros of the function. Use a graphing utility to verify your result.**

75. $f(x) = x^2(4 - x^2)$

76. $f(x) = \sqrt[3]{x}$

77. $f(x) = 2\sqrt[6]{x}$

78. $f(x) = 4x^4 - 3x^2$

Writing Functions **In Exercises 79–82, write an equation for a function that has the given graph.**

79. Line segment connecting $(-2, 4)$ and $(0, -6)$

80. Line segment connecting $(3, 1)$ and $(5, 8)$

81. The bottom half of the parabola $x + y^2 = 0$

82. The bottom half of the circle $x^2 + y^2 = 36$

Sketching a Graph **In Exercises 83–86, sketch a possible graph of the situation.**

83. The speed of an airplane as a function of time during a 5-hour flight

84. The height of a baseball as a function of horizontal distance during a home run

85. A student commutes 15 miles to attend college. After driving for a few minutes, she remembers that a term paper that is due has been forgotten. Driving faster than usual, she returns home, picks up the paper, and once again starts toward school. Consider the student's distance from home as a function of time.

86. A person buys a new car and keeps it for 6 years. During year 4, he buys several expensive upgrades. Consider the value of the car as a function of time.

87. **Domain** Find the value of c such that the domain of

$$f(x) = \sqrt{c - x^2}$$

is $[-5, 5]$.

88. **Domain** Find all values of c such that the domain of

$$f(x) = \frac{x + 3}{x^2 + 3cx + 6}$$

is the set of all real numbers.

EXPLORING CONCEPTS

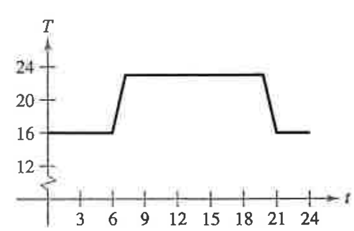
89. **One-to-One Functions** Can the graph of a one-to-one function intersect a horizontal line more than once? Explain.

90. **Composite Functions** Give an example of functions f and g such that $f \circ g = g \circ f$ and $f(x) \neq g(x)$.

91. **Polynomial Functions** Does the degree of a polynomial function determine whether the function is even or odd? Explain.

92. **Think About It** Determine whether the function $f(x) = 0$ is even, odd, both, or neither. Explain.

93. **Graphical Reasoning** An electronically controlled thermostat is programmed to lower the temperature during the night automatically (see figure). The temperature T in degrees Celsius is given in terms of t, the time in hours on a 24-hour clock.

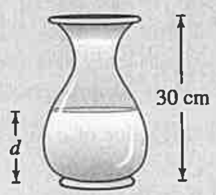


(a) Approximate $T(4)$ and $T(15)$.

(b) The thermostat is reprogrammed to produce a temperature $H(t) = T(t - 1)$. How does this change the temperature? Explain.

(c) The thermostat is reprogrammed to produce a temperature $H(t) = T(t) - 1$. How does this change the temperature? Explain.

94. HOW DO YOU SEE IT? Water runs into a vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds. Use this information and the shape of the vase shown to answer the questions when d is the depth of the water in centimeters and t is the time in seconds (see figure).



(a) Explain why d is a function of t.

(b) Determine the domain and range of the function.

(c) Sketch a possible graph of the function.

(d) Use the graph in part (c) to approximate $d(4)$. What does this represent?

95. Automobile Aerodynamics The horsepower H required to overcome wind drag on a certain automobile is

$$H(x) = 0.00004636x^3$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph H.

(b) Rewrite H so that x represents the speed in kilometers per hour. [*Hint:* Find $H(x/1.6)$.]

96. Writing Use a graphing utility to graph the polynomial functions

$$p_1(x) = x^3 - x + 1 \quad \text{and} \quad p_2(x) = x^3 - x.$$

How many zeros does each function have? Is there a cubic polynomial that has no zeros? Explain.

97. Proof Prove that the function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x$$

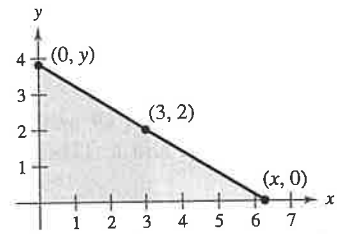
98. Proof Prove that the function is even.

$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$$

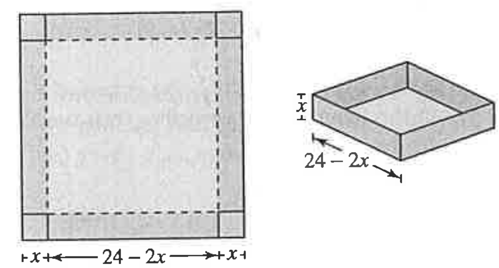
99. Proof Prove that the product of two even (or two odd) functions is even.

100. Proof Prove that the product of an odd function and an even function is odd.

101. Length A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(3, 2)$ (see figure). Write the length L of the hypotenuse as a function of x.



102. Volume An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).



(a) Write the volume V as a function of x, the length of the corner squares. What is the domain of the function?

(b) Use a graphing utility to graph the volume function and approximate the dimensions of the box that yield a maximum volume.

True or False? **In Exercises 103–108, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. If $f(a) = f(b)$, then $a = b$.

104. A vertical line can intersect the graph of a function at most once.

105. If $f(x) = f(-x)$ for all x in the domain of f, then the graph of f is symmetric with respect to the y-axis.

106. If f is a function, then $f(ax) = af(x)$.

107. The graph of a function of x cannot have symmetry with respect to the x-axis.

108. If the domain of a function consists of a single number, then its range must also consist of only one number.

PUTNAM EXAM CHALLENGE

109. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

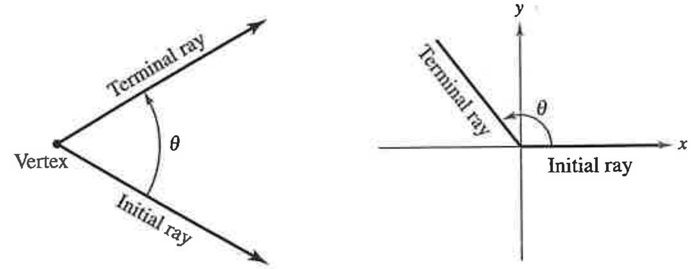
110. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.

P.4 Review of Trigonometric Functions

- **Describe angles and use degree measure.**
- **Use radian measure.**
- **Understand the definitions of the six trigonometric functions.**
- **Evaluate trigonometric functions.**
- **Solve trigonometric equations.**
- **Graph trigonometric functions.**

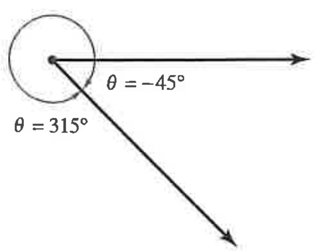
Angles and Degree Measure

An **angle** has three parts: an **initial ray** (or side), a **terminal ray,** and a **vertex** (the point of intersection of the two rays), as shown in Figure P.32(a). An angle is in **standard position** when its initial ray coincides with the positive x-axis and its vertex is at the origin, as shown in Figure P.32(b).



(a) Angle **(b)** Angle in standard position
Figure P.32

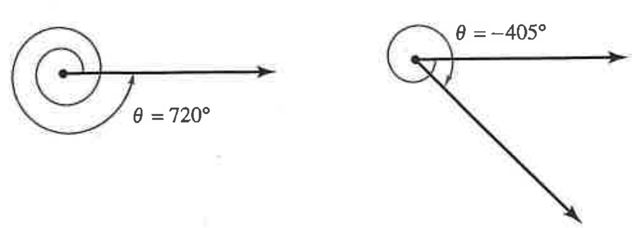
It is assumed that you are familiar with the degree measure of an angle.* It is common practice to use θ (the lowercase Greek letter theta) to represent both an angle and its measure. Angles between 0° and 90° are **acute,** and angles between 90° and 180° are **obtuse.**



Coterminal angles
Figure P.33

Positive angles are measured *counterclockwise*, and negative angles are measured *clockwise*. For instance, Figure P.33 shows an angle whose measure is $-45°$. You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved. For example, Figure P.33 shows that the angle measuring $-45°$ has the same terminal ray as the angle measuring 315°. Such angles are **coterminal.** In general, if θ is any angle, then $\theta + n(360)$, n is a nonzero integer, is coterminal with θ.

An angle that is larger than 360° is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure P.34(a). You can form an angle whose measure is less than $-360°$ by revolving a terminal ray more than one full revolution clockwise, as shown in Figure P.34(b).



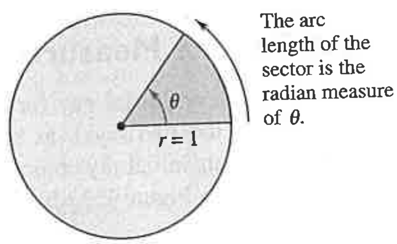
(a) An angle whose measure is greater than 360° **(b)** An angle whose measure is less than $-360°$
Figure P.34

*For a more complete review of trigonometry, see *Precalculus,* 10th edition, or *Precalculus: Real Mathematics, Real People,* 7th edition, both by Ron Larson (Boston, Massachusetts: Brooks/Cole, Cengage Learning, 2018 and 2016, respectively).

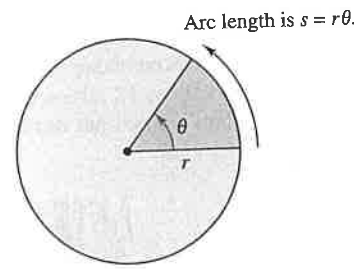
Radian Measure

To assign a radian measure to an angle θ, consider θ to be a central angle of a circle of radius 1, as shown in Figure P.35. The **radian measure** of θ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a **unit circle** (of radius 1) is 2π. This implies that the radian measure of an angle measuring 360° is 2π. In other words, $360° = 2\pi$ radians.

Using radian measure for θ, the length s of a circular arc of radius r is $s = r\theta$, as shown in Figure P.36.

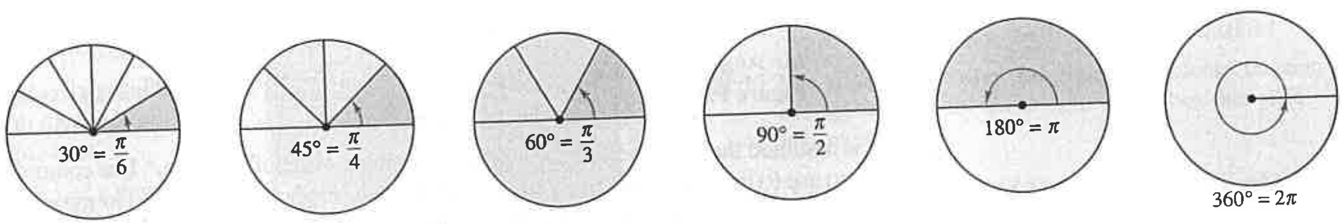


Unit circle
Figure P.35



Circle of radius r
Figure P.36

You should know the conversions of the common angles shown in Figure P.37. For other angles, use the fact that 180° is equal to π radians.



Radian and degree measures for several common angles
Figure P.37

EXAMPLE 1 **Conversions Between Degrees and Radians**

a. $40° = (40\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = \dfrac{2\pi}{9}$ radian

b. $540° = (540\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = 3\pi$ radians

c. $-270° = (-270\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = -\dfrac{3\pi}{2}$ radians

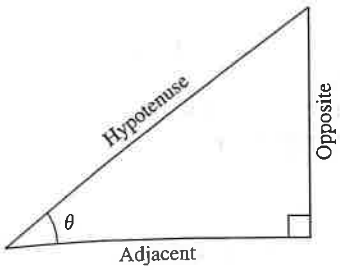
d. $-\dfrac{\pi}{2}$ radians $= \left(-\dfrac{\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = -90°$

e. 2 radians $= (2\ \cancel{\text{rad}})\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = \left(\dfrac{360}{\pi}\right)^{\circ} \approx 114.59°$

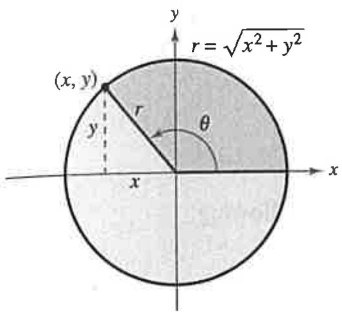
f. $\dfrac{9\pi}{2}$ radians $= \left(\dfrac{9\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = 810°$ ■

TECHNOLOGY Most graphing utilities have both *degree* and *radian* modes. You should learn how to use your graphing utility to convert from degrees to radians, and vice versa. Use a graphing utility to verify the results of Example 1.

The Trigonometric Functions



Sides of a right triangle
Figure P.38



An angle in standard position
Figure P.39

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal ray of an angle in standard position. The six trigonometric functions, **sine, cosine, tangent, cotangent, secant,** and **cosecant** (abbreviated as sin, cos, tan, cot, sec, and csc, respectively), are defined below from both viewpoints.

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$ (see Figure P.38)

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Circular function definitions, where θ is any angle (see Figure P.39)

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

$$\csc\theta = \frac{r}{y},\ y \neq 0 \qquad \sec\theta = \frac{r}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

The trigonometric identities listed below are direct consequences of the definitions. [Note that ϕ is the lowercase Greek letter phi and $\sin^2\theta$ is used to represent $(\sin\theta)^2$.]

TRIGONOMETRIC IDENTITIES

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Even/Odd Identities

$\sin(-\theta) = -\sin\theta$ $\qquad$ $\csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta$ $\qquad$ $\sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\cot(-\theta) = -\cot\theta$

Sum and Difference Formulas

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Power-Reducing Formulas

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Double-Angle Formulas

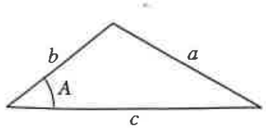
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions: (1) decimal approximations with a graphing utility and (2) exact evaluations using trigonometric identities and formulas from geometry. When using a graphing utility to evaluate a trigonometric function, remember to set the graphing utility to the appropriate mode—*degree* mode or *radian* mode.

EXAMPLE 2 Exact Evaluation of Trigonometric Functions

Evaluate the sine, cosine, and tangent of $\pi/3$.

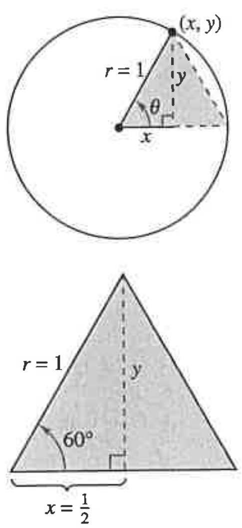


Figure P.40

Solution Because $60° = \pi/3$ radians, you can draw an equilateral triangle with sides of length 1 and θ as one of its angles, as shown in Figure P.40. Because the altitude of this triangle bisects its base, you know that $x = \frac{1}{2}$. Using the Pythagorean Theorem, you obtain

$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Now, knowing the values of x, y, and r, you can write the following.

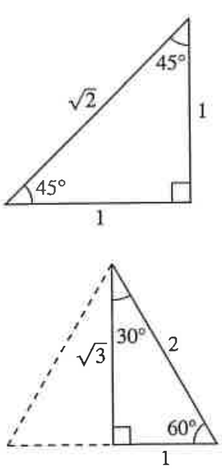
$$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Note that all angles in this text are measured in radians unless stated otherwise. For example, when sin 3 is written, the sine of 3 radians is meant, and when sin 3° is written, the sine of 3 degrees is meant.

The degree and radian measures of several common angles are shown in the table below, along with the corresponding values of the sine, cosine, and tangent (see Figure P.41).



Common angles
Figure P.41

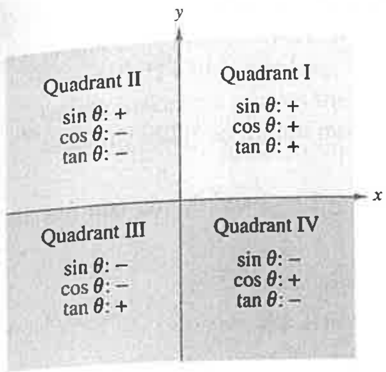
Trigonometric Values of Common Angles

θ (degrees)	0°	30°	45°	60°	90°	180°	270°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined	0	Undefined

EXAMPLE 3 Using Trigonometric Identities

a. $\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ $\qquad \sin(-\theta) = -\sin\theta$

b. $\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2$ $\qquad \sec\theta = \frac{1}{\cos\theta}$



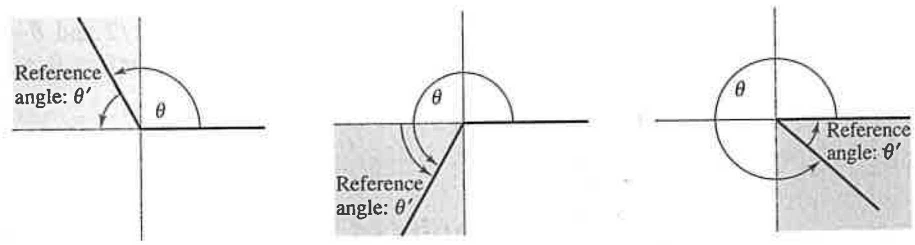
Quadrant signs for trigonometric functions
Figure P.42

The quadrant signs for the sine, cosine, and tangent functions are shown in Figure P.42. To extend the use of the table on the preceding page to angles in quadrants other than the first quadrant, you can use the concept of a **reference angle** (see Figure P.43), with the appropriate quadrant sign. For instance, the reference angle for $3\pi/4$ is $\pi/4$, and because the sine is positive in Quadrant II, you can write

$$\sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, because the reference angle for 330° is 30°, and the tangent is negative in Quadrant IV, you can write

$$\tan 330° = -\tan 30° = -\frac{\sqrt{3}}{3}.$$



Quadrant II
$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

Quadrant III
$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

Quadrant IV
$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

Figure P.43

Solving Trigonometric Equations

How would you solve the equation $\sin\theta = 0$? You know that $\theta = 0$ is one solution, but this is not the only solution. Any one of the following values of θ is also a solution.

$$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

You can write this infinite solution set as $\{n\pi\colon\ n \text{ is an integer}\}$.

EXAMPLE 4 Solving a Trigonometric Equation

Solve the equation $\sin\theta = -\dfrac{\sqrt{3}}{2}$.

Solution To solve the equation, you should consider that the sine function is negative in Quadrants III and IV and that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

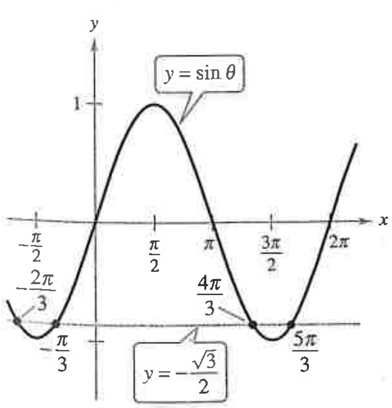
So, you are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$. In the interval $[0, 2\pi]$, the two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad\text{and}\quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

By adding integer multiples of 2π to each of these solutions, you obtain the following general solution.

$$\theta = \frac{4\pi}{3} + 2n\pi \quad\text{or}\quad \theta = \frac{5\pi}{3} + 2n\pi, \quad\text{where } n \text{ is an integer.}$$

See Figure P.44.



Solution points of $\sin\theta = -\dfrac{\sqrt{3}}{2}$
Figure P.44

EXAMPLE 5 Solving a Trigonometric Equation

REMARK Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 5, $\cos 2\theta$ means $\cos(2\theta)$.

Solve

$$\cos 2\theta = 2 - 3 \sin \theta$$

where $0 \le \theta \le 2\pi$.

Solution Using the double-angle formula $\cos 2\theta = 1 - 2\sin^2 \theta$, you can rewrite the equation as follows.

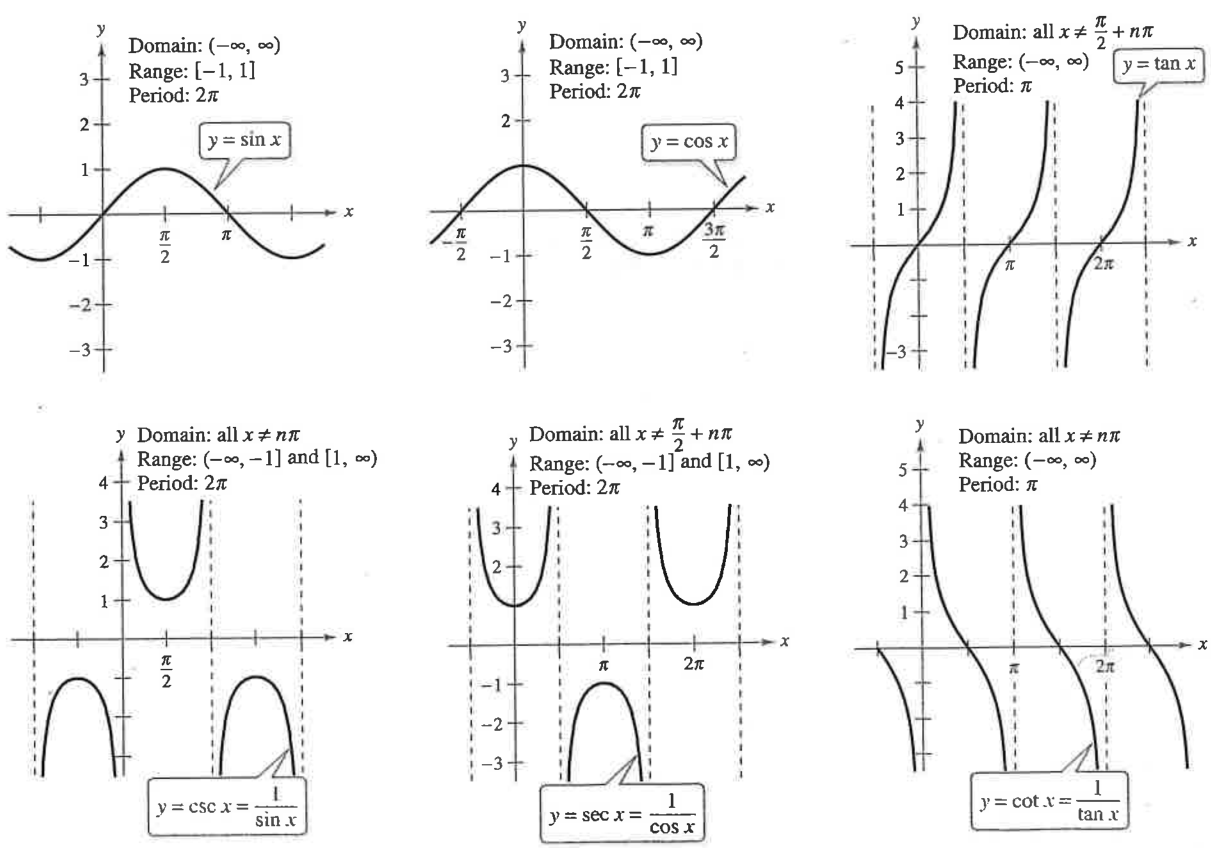
$\cos 2\theta = 2 - 3\sin\theta$	Write original equation.
$1 - 2\sin^2\theta = 2 - 3\sin\theta$	Double-angle formula
$0 = 2\sin^2\theta - 3\sin\theta + 1$	Quadratic form
$0 = (2\sin\theta - 1)(\sin\theta - 1)$	Factor.

If $2\sin\theta - 1 = 0$, then $\sin\theta = 1/2$ and $\theta = \pi/6$ or $\theta = 5\pi/6$. If $\sin\theta - 1 = 0$, then $\sin\theta = 1$ and $\theta = \pi/2$. So, for $0 \le \theta \le 2\pi$, the solutions are

$$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \text{or} \quad \frac{\pi}{2}.$$

Graphs of Trigonometric Functions

A function f is **periodic** when there exists a positive real number p such that $f(x + p) = f(x)$ for all x in the domain of f. The least such positive value of p is the **period** of f. The sine, cosine, secant, and cosecant functions each have a period of 2π, and the other two trigonometric functions, tangent and cotangent, have a period of π, as shown in Figure P.45.



The graphs of the six trigonometric functions
Figure P.45

▷ **TECHNOLOGY** To produce the graphs shown in Figure P.45 with a graphing utility, make sure you set the graphing utility to *radian* mode.

Note in Figure P.45 that the maximum value of $\sin x$ and $\cos x$ is 1 and the minimum value is -1. The graphs of the functions $y = a \sin bx$ and $y = a \cos bx$ oscillate between $-a$ and a, and so have an **amplitude** of $|a|$. Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a \sin bx$ and $y = a \cos bx$ each have a period of $2\pi/|b|$. The table below summarizes the amplitudes and periods of some types of trigonometric functions.

Function	Period	Amplitude
$y = a \sin bx$ or $y = a \cos bx$	$\frac{2\pi}{\lvert b \rvert}$	$\lvert a \rvert$
$y = a \tan bx$ or $y = a \cot bx$	$\frac{\pi}{\lvert b \rvert}$	Not applicable
$y = a \sec bx$ or $y = a \csc bx$	$\frac{2\pi}{\lvert b \rvert}$	Not applicable

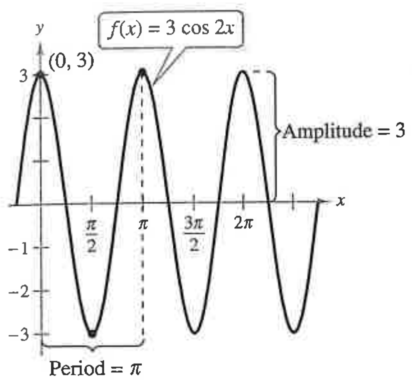


Figure P.46

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Sketch the graph of $f(x) = 3 \cos 2x$.

Solution The graph of $f(x) = 3 \cos 2x$ has an amplitude of 3 and a period of $2\pi/2 = \pi$. Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

Maximum: $(0, 3)$

Minimum: $\left(\frac{\pi}{2}, -3\right)$

Maximum: $(\pi, 3)$

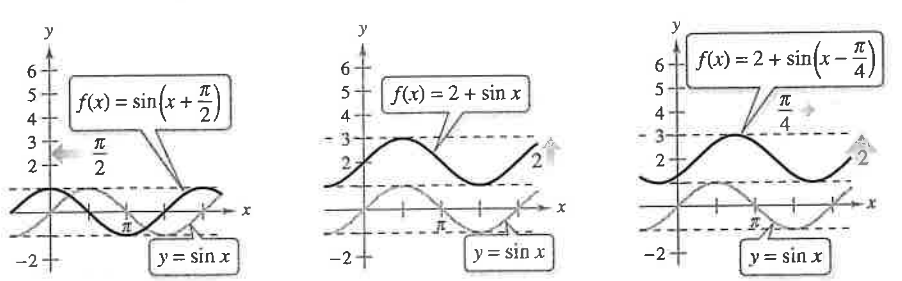
By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure P.46.

EXAMPLE 7 Shifts of Graphs of Trigonometric Functions

a. To sketch the graph of $f(x) = \sin(x + \pi/2)$, shift the graph of $y = \sin x$ to the left $\pi/2$ units, as shown in Figure P.47(a).

b. To sketch the graph of $f(x) = 2 + \sin x$, shift the graph of $y = \sin x$ upward two units, as shown in Figure P.47(b).

c. To sketch the graph of $f(x) = 2 + \sin(x - \pi/4)$, shift the graph of $y = \sin x$ upward two units and to the right $\pi/4$ units, as shown in Figure P.47(c).



(a) Horizontal shift to the left (b) Vertical shift upward (c) Horizontal and vertical shifts

Transformations of the graph of $y = \sin x$

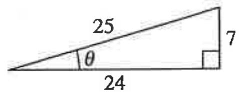
Figure P.47

P.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

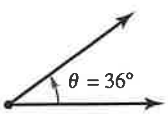
1. **Coterminal Angles** Explain how to find coterminal angles in degrees.
2. **Degrees to Radians** Explain how to convert from degrees to radians.
3. **Trigonometric Functions** Find $\sin \theta$, $\cos \theta$, and $\tan \theta$.



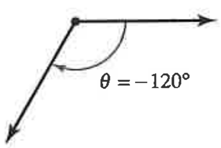
4. **Characteristics of a Graph** In your own words, describe the meaning of *amplitude* and *period.*

Coterminal Angles in Degrees **In Exercises 5 and 6, determine two coterminal angles in degree measure (one positive and one negative) for each angle.**

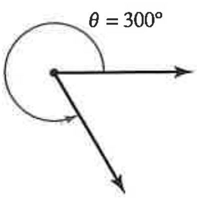
5. (a)



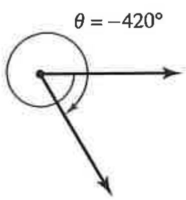
(b)



6. (a)

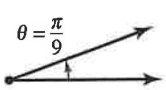


(b)

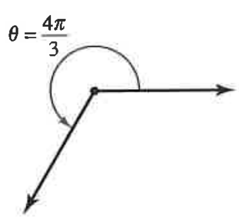


Coterminal Angles in Radians **In Exercises 7 and 8, determine two coterminal angles in radian measure (one positive and one negative) for each angle.**

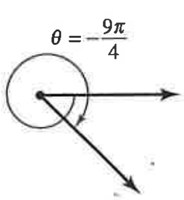
7. (a)



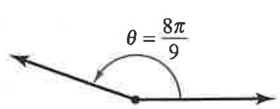
(b)



8. (a)



(b)



Degrees to Radians **In Exercises 9 and 10, convert the degree measure to radian measure as a multiple of π and as a decimal accurate to three decimal places.**

9. (a) 30° (b) 150° (c) 315° (d) 120°

10. (a) −20° (b) −240° (c) −270° (d) 144°

Radians to Degrees **In Exercises 11 and 12, convert the radian measure to degree measure.**

11. (a) $\frac{3\pi}{2}$ (b) $\frac{7\pi}{6}$ (c) $-\frac{7\pi}{12}$ (d) -2.367

12. (a) $\frac{7\pi}{3}$ (b) $-\frac{11\pi}{30}$ (c) $\frac{11\pi}{6}$ (d) 0.438

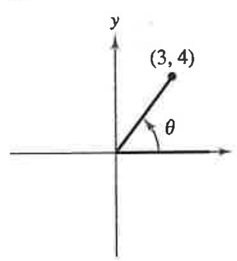
13. **Completing a Table** Let r represent the radius of a circle, θ the central angle (measured in radians), and s the length of the arc subtended by the angle. Use the relationship $s = r\theta$ to complete the table.

r	8 ft	15 in.	85 cm		
s	12 ft			96 in.	8642 mi
θ		1.6	$\frac{3\pi}{4}$	4	$\frac{2\pi}{3}$

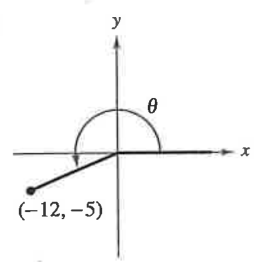
14. **Angular Speed** A car is moving at the rate of 50 miles per hour, and the diameter of its wheels is 2.5 feet.
    (a) Find the number of revolutions per minute that the wheels are rotating.
    (b) Find the angular speed of the wheels in radians per minute.

Evaluating Trigonometric Functions **In Exercises 15 and 16, evaluate the six trigonometric functions of the angle θ.**

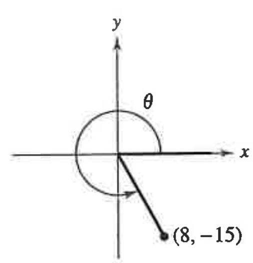
15. (a)



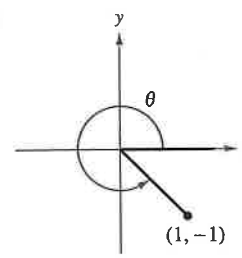
(b)



16. (a)



(b)



Evaluating Trigonometric Functions **In Exercises 17–20, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Then evaluate the other five trigonometric functions of θ.**

17. $\sin \theta = \frac{1}{2}$

18. $\sin \theta = \frac{1}{3}$

19. $\cos \theta = \frac{4}{5}$

20. $\sec \theta = \frac{13}{5}$

Evaluating Trigonometric Functions In Exercises 21–24, evaluate the sine, cosine, and tangent of each angle. Do not use a calculator.

21. (a) $60°$ (b) $120°$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$

22. (a) $-30°$ (b) $150°$ (c) $-\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

23. (a) $225°$ (b) $-225°$ (c) $\frac{5\pi}{3}$ (d) $\frac{11\pi}{6}$

24. (a) $750°$ (b) $510°$ (c) $\frac{10\pi}{3}$ (d) $\frac{17\pi}{3}$

Evaluating Trigonometric Functions Using Technology In Exercises 25–28, use a calculator to evaluate each trigonometric function. Round your answers to four decimal places.

25. (a) $\sin 10°$
 (b) $\csc 10°$

26. (a) $\sec 225°$
 (b) $\sec 135°$

27. (a) $\tan \frac{\pi}{9}$
 (b) $\tan \frac{10\pi}{9}$

28. (a) $\cot(1.35)$
 (b) $\tan(1.35)$

Determining a Quadrant In Exercises 29 and 30, determine the quadrant in which θ lies.

29. (a) $\sin \theta < 0$ and $\cos \theta < 0$
 (b) $\sec \theta > 0$ and $\cot \theta < 0$

30. (a) $\sin \theta > 0$ and $\cos \theta < 0$
 (b) $\csc \theta < 0$ and $\tan \theta > 0$

Solving a Trigonometric Equation In Exercises 31–34, find two solutions of each equation. Give your answers in radians ($0 \le \theta \le 2\pi$). Do not use a calculator.

31. (a) $\cos \theta = \frac{\sqrt{2}}{2}$
 (b) $\cos \theta = -\frac{\sqrt{2}}{2}$

32. (a) $\sec \theta = 2$
 (b) $\sec \theta = -2$

33. (a) $\tan \theta = 1$
 (b) $\cot \theta = -\sqrt{3}$

34. (a) $\sin \theta = \frac{\sqrt{3}}{2}$
 (b) $\sin \theta = -\frac{\sqrt{3}}{2}$

Solving a Trigonometric Equation In Exercises 35–42, solve the equation for θ, where $0 \le \theta \le 2\pi$.

35. $2\sin^2\theta = 1$

36. $\tan^2\theta = 3$

37. $\tan^2\theta - \tan\theta = 0$

38. $2\cos^2\theta - \cos\theta = 1$

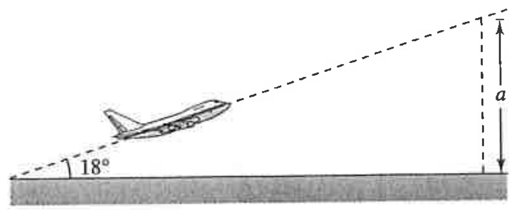
39. $\sec\theta\csc\theta = 2\csc\theta$

40. $\sin\theta = \cos\theta$

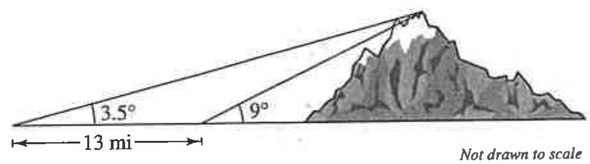
41. $\cos^2\theta + \sin\theta = 1$

42. $\cos\frac{\theta}{2} - \cos\theta = 1$

43. **Airplane Ascent** An airplane leaves the runway climbing at an angle of 18° with a speed of 275 feet per second (see figure). Find the altitude a of the plane after 1 minute.

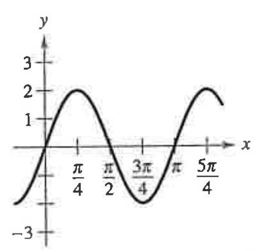


44. **Height of a Mountain** While traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9°. Approximate the height of the mountain.

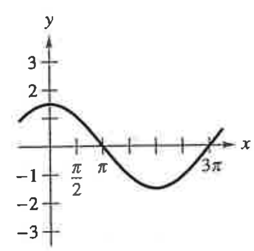


Period and Amplitude In Exercises 45–48, determine the period and amplitude of each function.

45. $y = 2\sin 2x$



46. $y = \frac{3}{2}\cos\frac{x}{2}$



47. $y = -3\sin 4\pi x$

48. $y = \frac{2}{3}\cos\frac{\pi x}{10}$

Period In Exercises 49–52, find the period of the function.

49. $y = 5\tan 2x$

50. $y = 7\tan 2\pi x$

51. $y = \sec 5x$

52. $y = \csc 4x$

Writing In Exercises 53 and 54, use a graphing utility to graph each function f in the same viewing window for $c = -2$, $c = -1$, $c = 1$, and $c = 2$. Give a written description of the change in the graph caused by changing c.

53. (a) $f(x) = c\sin x$
 (b) $f(x) = \cos(cx)$
 (c) $f(x) = \cos(\pi x - c)$

54. (a) $f(x) = \sin x + c$
 (b) $f(x) = -\sin(2\pi x - c)$
 (c) $f(x) = c\cos x$

Sketching the Graph of a Trigonometric Function **In Exercises 55–66, sketch the graph of the function.**

55. $y = \sin \frac{x}{2}$

56. $y = 2 \cos 2x$

57. $y = -\sin \frac{2\pi x}{3}$

58. $y = 2 \tan x$

59. $y = \csc \frac{x}{2}$

60. $y = \tan 2x$

61. $y = 2 \sec 2x$

62. $y = \csc 2\pi x$

63. $y = \sin(x + \pi)$

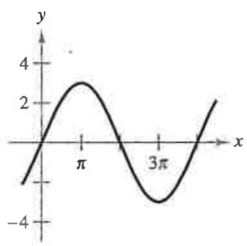
64. $y = \cos\left(x - \frac{\pi}{3}\right)$

65. $y = 1 + \cos\left(x - \frac{\pi}{2}\right)$

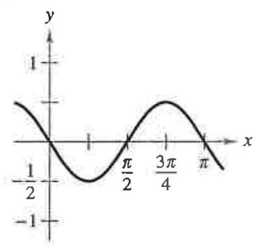
66. $y = 1 + \sin\left(x + \frac{\pi}{2}\right)$

Graphical Reasoning **In Exercises 67 and 68, find *a*, *b*, and *c* such that the graph of the function matches the graph in the figure.**

67. $y = a \cos(bx - c)$



68. $y = a \sin(bx - c)$



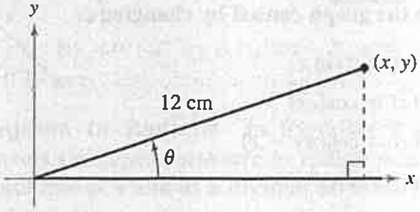
EXPLORING CONCEPTS

69. **Think About It** You are given the value of $\tan \theta$. Is it possible to find the value of $\sec \theta$ without finding the measure of θ? Explain.

70. **Restricted Domain** Explain how to restrict the domain of the sine function so that it becomes a one-to-one function.

71. **Think About It** How do the ranges of the cosine function and the secant function compare?

72. **HOW DO YOU SEE IT?** Consider an angle in standard position with $r = 12$ centimeters, as shown in the figure. Describe the changes in the values of x, y, $\sin \theta$, $\cos \theta$, and $\tan \theta$ as θ increases continually from 0° to 90°.



73. **Think About It** Sketch the graphs of

$$f(x) = \sin x, \quad g(x) = |\sin x|, \quad \text{and} \quad h(x) = \sin(|x|).$$

In general, how are the graphs of $|f(x)|$ and $f(|x|)$ related to the graph of f?

74. **Ferris Wheel** The model for the height h of a Ferris wheel car is

$$h = 51 + 50 \sin 8\pi t$$

where t is measured in minutes. (The Ferris wheel has a radius of 50 feet.) This model yields a height of 51 feet when $t = 0$. Alter the model so that the height of the car is 1 foot when $t = 0$.

75. **Sales** The monthly sales S (in thousands of units) of a seasonal product are modeled by

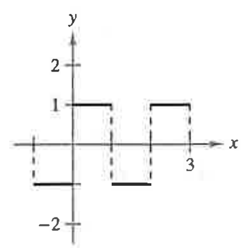
$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January. Use a graphing utility to graph the model for S and determine the months when sales exceed 75,000 units.

76. **Pattern Recognition** Use a graphing utility to compare the graph of

$$f(x) = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3} \sin 3\pi x\right)$$

with the given graph. Try to improve the approximation by adding a term to $f(x)$. Use a graphing utility to verify that your new approximation is better than the original. Can you find other terms to add to make the approximation even better? What is the pattern? (*Hint:* Use sine terms.)



True or False? **In Exercises 77–80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. A measurement of 4 radians corresponds to two complete revolutions from the initial side to the terminal side of an angle.

78. Amplitude is always positive.

79. The function $y = \frac{1}{2} \sin 2x$ has an amplitude that is twice that of the function $y = \sin x$.

80. The function $y = 3 \cos(x/3)$ has a period that is three times that of the function $y = \cos x$.

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